GLOBAL WAVE FRONT SET OF MODULATION SPACE TYPES

SANDRO CORIASCO, KAROLINE JOHANSSON, AND JOACHIM TOFT

Abstract. We introduce global wave-front sets $\mathrm{WF}_{M(\omega,\mathscr{B})}(f)$, $f \in \mathscr{S}'(\mathbf{R}^d)$, with respect to the modulation spaces $M(\omega,\mathscr{B})$, where ω is an appropriate weight function and $\mathscr{B}$ is a translation invariant Banach function space. We show that the standard properties for known notions of wave-front set extend to $\mathrm{WF}_{M(\omega,\mathscr{B})}(f)$. In particular, we prove that microlocality and microellipticity hold for a class of globally defined pseudo-differential operators $\mathrm{Op}(a)$.

0. Introduction

In this paper we introduce wave-front sets for tempered distributions with respect to general modulation spaces $M(\omega,\mathscr{B})$, parameterized with the translation invariant Banach function space $\mathscr{B}$ and the weight function ω.

Our approach links the "local" analysis carried on in [7, 38], with the "global" analogue treated, e.g., in [8]. In the latter reference, wave-front sets are used to describe propagation of singularities for a class of hyperbolic Cauchy problems, associated with linear operators having coefficients growing (at most) polynomially with respect to the space variable (cf., e.g., Cordes [5], Coriasco [6], Coriasco and Rodino [9], Cappiello and Rodino [4], Melrose [33], Parenti [35]). In view of such hypotheses, the behaviour at infinity of tempered distributions, belonging to suitable weighted Sobolev spaces, can be controlled: under the action of the global pseudo-differential operators with symbol in the SG classes, in addition to the loss of smoothness, a loss of decay is in general present.

In particular, we recover the microlocality and microellipticity properties that hold for wave-front sets of Sobolev type introduced by Hörmander [31], and classical wave-front sets with respect to smoothness (cf. Sections 8.1 and 8.2 in [30]), as well as for wave-front sets of Banach function types in [7], and $\mathscr{S}$ and H^{s_1,s_2} wave-front sets in [8] and [33].

In analogy with the definitions in [8], we introduce a triple

$$(\mathrm{WF}^{\psi}_{M(\omega,\mathscr{B})}(f), \mathrm{WF}^{e}_{M(\omega,\mathscr{B})}(f), \mathrm{WF}^{\psi e}_{M(\omega,\mathscr{B})}(f))$$

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of wave-front sets with respect to modulation spaces, when f is a tempered distribution. The notation $\mathrm{WF}_{M(\omega,\mathscr{B})}(f)$ is used for their union, and it is called the global wave-front set. Compared with the wave-front set defined in [7], the three components of $\mathrm{WF}_{M(\omega,\mathscr{B})}(f)$ describe different properties, like the three subsets of the $\mathscr{S}$ wave-front set in [8].

Roughly speaking, $\mathrm{WF}^{\psi}_{M(\omega,\mathscr{B})}(f)$, $\mathrm{WF}^{e}_{M(\omega,\mathscr{B})}(f)$ and $\mathrm{WF}^{\psi e}_{M(\omega,\mathscr{B})}(f)$ give information about the local smoothness of f, the size of f at infinity, and about oscillations at infinity, respectively. In particular, the union $\mathrm{WF}^{e}_{M(\omega,\mathscr{B})}(f) \cup \mathrm{WF}^{\psi e}_{M(\omega,\mathscr{B})}(f)$, the so called "exit component", describes the behavior "at infinity" of f. Moreover, $f \in M(\omega,\mathscr{B})$ if and only if $\mathrm{WF}_{M(\omega,\mathscr{B})}(f) = \emptyset$.

The modulation spaces were introduced in [12] by Feichtinger and were further developed and generalized in [14–16, 20]. The modulation space $M(\omega,\mathscr{B})$, where ω denotes a weight on phase (or time-frequency shift) space $\mathbf{R}^{2d}$, appears as the set of tempered (ultra-)distributions whose short-time Fourier transform belongs to the weighted space $\mathscr{B}(\omega)$, which is assumed to be a (translation) invariant Banach function spaces (cf. [12–15]).

The operators involved in the sequel are a generalisation of the so-called SG operators (see, e.g., Cordes [5], Egorov and Schulze [11], Schrohe [41], Melrose [33], and Parenti [35]). The corresponding symbols a belong to the classes denoted by $\mathrm{SG}^{(\omega)}_{r,\rho}(\mathbf{R}^{2d})$ or $\mathrm{SG}^{(\omega)}_{r,\rho}$, for $r,\rho \geq 0$ and a weight function ω, if they satisfy the global estimates

$$|D^{\alpha}_{x} D^{\beta}_{\xi} a(x,\xi)| \leq C_{\alpha,\beta} \langle x\rangle^{-r|\alpha|} \langle \xi\rangle^{-\rho|\beta|} \omega(x,\xi), \qquad x,\xi \in \mathbf{R}^{d}, \tag{0.1}$$

for suitable constants $C_{\alpha,\beta} \geq 0$. Such globally defined operators turn out to be continuous on the modulation spaces $M(\omega,\mathscr{B})$ (see Theorem 3.2 in [53]).

The paper is organized as follows. In Section 1 we recall the definition and basic properties of pseudo-differential operators, translation invariant Banach function spaces and modulation spaces. Here we also define three types of sets of characteristic points and some properties for them. One of these characteristic sets coincide with the one defined in [7], but all of them differ by the corresponding ones considered in [8] and [4].

In Section 2 we define the wave-front sets of modulation space types $\mathrm{WF}_{M(\omega,\mathscr{B})}(f)$. Furthermore, prove that $\mathrm{WF}_{M(\omega,\mathscr{B})}(f)$ and $\mathrm{WF}_{\mathscr{F}\mathscr{B}_0(\omega)}(f)$ coincide with the wave-front sets defined in [7], when $M(\omega,\mathscr{B})$ is locally the same as the Fourier BF-space $\mathscr{F}\mathscr{B}_0(\omega)$. The remaining part of the section is devoted to the proof of a relation between the wave-front sets of modulation space types and the sets of characteristic points, and of the fact that $\mathrm{WF}_{M(\omega,\mathscr{B})}(f)$ is empty if and only if the temperate distribution f belongs to $M(\omega,\mathscr{B})$.

Sections 3 and 4 are devoted to mapping properties for pseudo-differential operators in the context of these wave-front sets. The properties are given and proved in Section 3, and similar properties are stated in Section 4 for wave-front sets of superposition types. Finally, we have included an Appendix, where we prove some of the properties for the sets of characteristic points stated in Section 1.

1. Preliminaries

In this section we recall some notation and basic results. The proofs are in general omitted. In what follows we let Γ denote an open cone in $\mathbf{R}^d \setminus 0$. If $\xi \in \mathbf{R}^d \setminus 0$ is fixed, then an open cone which contains ξ is sometimes denoted by Γ_ξ.

Let ω and v be positive measurable functions on $\mathbf{R}^d$. Then ω is called v-moderate if

$$\omega(x+y) \le C\omega(x)v(y) \tag{1.1}$$

for some constant C which is independent of $x, y \in \mathbf{R}^d$. If v in (1.1) can be chosen as a polynomial, then ω is called polynomially moderated. We let $\mathscr{P}(\mathbf{R}^d)$ be the set of all polynomially moderated functions on $\mathbf{R}^d$. If $\omega(x,\xi) \in \mathscr{P}(\mathbf{R}^{2d})$ is constant with respect to the x-variable or the ξ-variable, then we sometimes write $\omega(\xi)$, respectively $\omega(x)$, instead of $\omega(x,\xi)$. In this case we consider ω as an element in $\mathscr{P}(\mathbf{R}^{2d})$ or in $\mathscr{P}(\mathbf{R}^d)$ depending on the situation. We say that v is submultiplicative if (1.1) holds for $\omega = v$. For conveniency we assume that all submultiplicative weights are even.

We also need to consider classes of weight functions, related to $\mathscr{P}$. More precisely, we let $\mathscr{P}_0(\mathbf{R}^d)$ be the set of all $\omega \in \mathscr{P}(\mathbf{R}^d)\bigcap C^\infty(\mathbf{R}^d)$ such that $\partial^\alpha\omega/\omega \in L^\infty$ for all multi-indices α. By Lemma 1.2 in [50] it follows that for each $\omega \in \mathscr{P}(\mathbf{R}^d)$, there is an element $\omega_0 \in \mathscr{P}_0(\mathbf{R}^d)$ which is equivalent to ω in the sense that

$$C^{-1}\omega_0 \le \omega \le C\omega_0, \tag{1.2}$$

for some positive constant C.

We need some more conditions for the weights in $\mathrm{SG}^{(\omega)}_{r,\rho}$ (cf. (0.1)). More precisely let $r,\rho \ge 0$. Then we let $\mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ be the set of all $\omega(x,\xi)$ in $\mathscr{P}(\mathbf{R}^{2d}) \cap C^\infty(\mathbf{R}^{2d})$ such that

$$\langle x\rangle^{r|\alpha|}\langle\xi\rangle^{\rho|\beta|}\frac{\partial_x^\alpha\partial_\xi^\beta\omega(x,\xi)}{\omega(x,\xi)} \in L^\infty(\mathbf{R}^{2d}),$$

for every multi-indices α and β. Note that $\mathscr{P}_{r,\rho}$ is different here compared to [7]. Note also that in contrast to $\mathscr{P}_0$, we do not have an equivalence between $\mathscr{P}_{r,\rho}$ and $\mathscr{P}$ when $r > 0$ or $\rho > 0$ in the sense of (1.2). On the other hand, if $s,t \in \mathbf{R}$ and $r,\rho \in [0,1]$, then $\mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ contains $\omega(x,\xi) = \langle x\rangle^t\langle\xi\rangle^s$, which gives one of the most important classes in the applications.

The Fourier transform $\mathscr{F}$ is the linear and continuous mapping on $\mathscr{S}'(\mathbf{R}^d)$ which takes the form

$$(\mathscr{F}f)(\xi) = \widehat{f}(\xi) \equiv (2\pi)^{-d/2} \int_{\mathbf{R}^d} f(x)e^{-i\langle x,\xi\rangle}\, dx$$

when $f \in L^1(\mathbf{R}^d)$. We recall that $\mathscr{F}$ is a homeomorphism on $\mathscr{S}'(\mathbf{R}^d)$ which restricts to a homeomorphism on $\mathscr{S}(\mathbf{R}^d)$ and to a unitary operator on $L^2(\mathbf{R}^d)$.

Next we define our Banach function spaces (BF-spaces) and present some useful properties.

Definition 1.1. Assume that $\mathscr{B} \subseteq L^1_{\mathrm{loc}}(\mathbf{R}^d)$ is a Banach space and that $v \in \mathscr{P}(\mathbf{R}^d)$ is submultiplicative. Then $\mathscr{B}$ is called a *(translation) invariant BF-space on* $\mathbf{R}^d$ (with respect to v), if there is a constant C such that the following conditions are fulfilled:

(1) $\mathscr{S}(\mathbf{R}^d) \subseteq \mathscr{B} \subseteq \mathscr{S}'(\mathbf{R}^d)$ (continuous embeddings);

(2) if $x \in \mathbf{R}^d$ and $f \in \mathscr{B}$, then $f(\cdot - x) \in \mathscr{B}$, and

$$\|f(\cdot - x)\|_{\mathscr{B}} \le Cv(x)\|f\|_{\mathscr{B}}; \tag{1.3}$$

(3) if $f, g \in L^1_{\mathrm{loc}}(\mathbf{R}^d)$ satisfy $g \in \mathscr{B}$ and $|f| \le |g|$ almost everywhere, then $f \in \mathscr{B}$ and

$$\|f\|_{\mathscr{B}} \le C\|g\|_{\mathscr{B}}.$$

Assume that $\mathscr{B}$ is a translation invariant BF-space. If $f \in \mathscr{B}$ and $h \in L^\infty$, then it follows from (3) in Definition 1.1 that $f \cdot h \in \mathscr{B}$ and

$$\|f \cdot h\|_{\mathscr{B}} \le C\|f\|_{\mathscr{B}}\|h\|_{L^\infty}. \tag{1.4}$$

Also let $\omega \in \mathscr{P}(\mathbf{R}^d)$. Then we let the Fourier BF-space $\mathscr{F}\mathscr{B}(\omega)$ be the set of all $f \in \mathscr{S}'(\mathbf{R}^d)$ such that $\xi \mapsto \widehat{f}(\xi)\omega(\xi)$ belongs to $\mathscr{B}$. It follows that $\mathscr{F}\mathscr{B}(\omega)$ is a Banach space under the norm

$$\|f\|_{\mathscr{F}\mathscr{B}(\omega)} \equiv \|\widehat{f}\,\omega\|_{\mathscr{B}}. \tag{1.5}$$

Remark 1.2. In many situations it is convenient to permit an x dependency for the weight ω in the definition of Fourier Banach spaces. More precisely, for each $\omega(x,\xi) \in \mathscr{P}(\mathbf{R}^{2d})$ and each translation invariant BF-space $\mathscr{B}$ on $\mathbf{R}^d$, we let $\mathscr{F}\mathscr{B}(\omega)$ be the set of all $f \in \mathscr{S}'(\mathbf{R}^d)$ such that

$$\|f\|_{\mathscr{F}\mathscr{B}(\omega)} = \|f\|_{\mathscr{F}\mathscr{B}(\omega),x} \equiv \|f\omega(x,\cdot)\|_{\mathscr{B}}$$

is finite. Since ω is v-moderate for some $v \in \mathscr{P}(\mathbf{R}^{2d})$ it follows that different choices of x give rise to equivalent norms. Therefore the condition $\|f\|_{\mathscr{F}\mathscr{B}(\omega)} < \infty$ is independent of x, and it follows that $\mathscr{F}\mathscr{B}(\omega)$ is independent of x although $\|\cdot\|_{\mathscr{F}\mathscr{B}(\omega)}$ might depend on x.

In order to define modulation spaces, we need to consider a local Fourier transforms. First assume that $\phi \in \mathscr{S}(\mathbf{R}^d)$. Then the *short-time Fourier transform* of $f \in \mathscr{S}(\mathbf{R}^d)$ with respect to (the window function) ϕ is defined by

$$V_\phi f(x,\xi) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} f(y)\overline{\phi(y-x)} e^{-i\langle y,\xi\rangle}\, dy. \tag{1.6}$$

If instead $\phi \in \mathscr{S}'(\mathbf{R}^d)$ and $f \in \mathscr{S}'(\mathbf{R}^d)$, then the short-time Fourier transform of f with respect to ϕ is defined by

$$(V_\phi f) = \mathscr{F}_2 F, \quad \text{where} \quad F(x,y) = (f \otimes \overline{\phi})(y, y-x). \tag{1.6$'$}$$

Here $\mathscr{F}_2 F$ is the partial Fourier transform of $F(x,y) \in \mathscr{S}'(\mathbf{R}^{2d})$ with respect to the y-variable. The definition (1.6)$'$ makes sense, since the mappings $\mathscr{F}_2$ and $F \mapsto F \circ G$ with $G(x,y) = (y, y-x)$ are homeomorphisms on $\mathscr{S}'(\mathbf{R}^{2d})$.

In the following lemma we recall some general continuity properties of the short-time Fourier transform. We omit the proof since the result can be found, e. g., in [18].

Lemma 1.3. *Let T be the mapping from $\mathscr{S}'(\mathbf{R}^d) \times \mathscr{S}'(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^{2d})$ which is given by $T(f,\phi) = V_\phi f$, and assume that $f, \phi \in \mathscr{S}'(\mathbf{R}^d) \setminus 0$. Then the following is true:*

(1) *T restricts to a continuous map from $\mathscr{S}(\mathbf{R}^d) \times \mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}(\mathbf{R}^{2d})$. Furthermore, $V_\phi f \in \mathscr{S}(\mathbf{R}^{2d})$ if and only if $f, \phi \in \mathscr{S}(\mathbf{R}^d)$;*

(2) *T restricts to a continuous map from $L^2(\mathbf{R}^d) \times L^2(\mathbf{R}^d)$ to $L^2(\mathbf{R}^{2d})$. Furthermore, $V_\phi f \in L^2(\mathbf{R}^{2d})$ if and only if $f, \phi \in L^2(\mathbf{R}^d)$;*

(3) *T is a continuous map from $\mathscr{S}'(\mathbf{R}^d) \times \mathscr{S}'(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^{2d})$, and restricts to a continuous map from $\mathscr{S}'(\mathbf{R}^d) \times \mathscr{S}(\mathbf{R}^d)$ and from $\mathscr{S}(\mathbf{R}^d) \times \mathscr{S}'(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^{2d}) \bigcap C^\infty(\mathbf{R}^{2d})$.*

For Lemma 1.4 below, we recall some formulae for the short-time Fourier transform which are important in time-frequency analysis (see e. g. [21]). For this reason it is convenient to let $\widehat{*}$ be the twisted convolution on $L^1(\mathbf{R}^{2d})$, defined by the formula

$$(F \mathbin{\widehat{*}} G)(x,\xi) = (2\pi)^{-d/2} \iint F(x-y, \xi-\eta) G(y,\eta) e^{-i\langle x-y,\eta\rangle}\, dy d\eta.$$

By straight-forward computations it follows that $\widehat{*}$ restricts to a continuous multiplication on $\mathscr{S}(\mathbf{R}^{2d})$. Furthermore, the map $(F,G) \mapsto F \mathbin{\widehat{*}} G$ from $\mathscr{S}(\mathbf{R}^{2d}) \times \mathscr{S}(\mathbf{R}^{2d})$ to $\mathscr{S}(\mathbf{R}^{2d})$ extends uniquely to continuous mappings from $\mathscr{S}'(\mathbf{R}^{2d}) \times \mathscr{S}(\mathbf{R}^{2d})$ and $\mathscr{S}(\mathbf{R}^{2d}) \times \mathscr{S}'(\mathbf{R}^{2d})$ to $\mathscr{S}'(\mathbf{R}^{2d}) \bigcap C^\infty(\mathbf{R}^{2d})$.

The following lemma is a straight-forward consequence of Fourier's inversion formula. The proof is therefore omitted. Here and in what follows we let $\widetilde{f}(x) = \overline{f(-x)}$.

Lemma 1.4. *Assume that $f \in \mathscr{S}'(\mathbf{R}^d)$ and that $\phi, \phi_j \in \mathscr{S}(\mathbf{R}^d)$ for $j = 1, 2, 3$. Then the following is true:*

(1) $(V_\phi f)(x, \xi) = e^{-i\langle x, \xi\rangle}(V_{\widehat{\phi}} \widehat{f})(\xi, -x)$;

(2) $(V_{\phi_1} f)\widehat{*}(V_{\phi_2}\phi_3) = (\phi_3, \phi_1)_{L^2} \cdot V_{\phi_2} f$;

(3) $(V_{\phi_1\overline{\phi_2}} f)(x, \xi) = (2\pi)^{-d/2} \displaystyle\int_{\mathbf{R}^d} (V_{\phi_1} f)(x, \xi - \eta)\widehat{\phi}_2(\eta)e^{-i\langle x, \eta\rangle}\, d\eta$;

(4) $(V_{\phi_1 * \widetilde{\phi}_2} f)(x, \xi) = \displaystyle\int_{\mathbf{R}^d} (V_{\phi_1} f)(x - y, \xi)\phi_2(y)\, dy.$

Now assume that $\mathscr{B}$ is a translation invariant BF-space on $\mathbf{R}^{2d}$, with respect to $v \in \mathscr{P}(\mathbf{R}^{2d})$. Also let $\phi \in \mathscr{S}(\mathbf{R}^d)\backslash 0$ and that $\omega \in \mathscr{P}(\mathbf{R}^{2d})$ is such that ω is v-moderate. The modulation space $M(\omega, \mathscr{B})$ consists of all $f \in \mathscr{S}'(\mathbf{R}^d)$ such that $V_\phi f \cdot \omega \in \mathscr{B}$. We note that $M(\omega, \mathscr{B})$ is a Banach space with the norm

$$\|f\|_{M(\omega,\mathscr{B})} \equiv \|V_\phi f \omega\|_{\mathscr{B}} \tag{1.7}$$

(cf. [14]).

Remark 1.5. Assume that $p, q \in [1, \infty]$, and let $L_1^{p,q}(\mathbf{R}^{2d})$ and $L_2^{p,q}(\mathbf{R}^{2d})$ be the sets of all $F \in L^1_{\mathrm{loc}}(\mathbf{R}^{2d})$ such that

$$\|F\|_{L_1^{p,q}} \equiv \Big(\int \Big(\int |F(x, \xi)|^p\, dx \Big)^{q/p} d\xi \Big)^{1/q} < \infty$$

and

$$\|F\|_{L_2^{p,q}} \equiv \Big(\int \Big(\int |F(x, \xi)|^p\, d\xi \Big)^{q/p} dx \Big)^{1/q} < \infty.$$

Then $M(\omega, L_1^{p,q}(\mathbf{R}^{2d}))$ is equal to the usual modulation space $M^{p,q}_{(\omega)}(\mathbf{R}^d)$, and $M(\omega, L_2^{p,q}(\mathbf{R}^{2d}))$ is equal to the space $W^{p,q}_{(\omega)}(\mathbf{R}^d)$, related to Wiener-amalgam spaces.

For notational convenience we set $M^p_{(\omega)} = M^{p,p}_{(\omega)} = W^{p,p}_{(\omega)}$. Furthermore, if $\omega = 1$, then we write $M^{p,q}$, M^p and $W^{p,q}$ instead of $M^{p,q}_{(\omega)}$, $M^p_{(\omega)}$ and $W^{p,q}_{(\omega)}$ respectively.

In the following proposition we list some important properties for modulation spaces. We refer to [21] for the proof.

Proposition 1.6. *Let $\omega_0, v_0, v \in \mathscr{P}(\mathbf{R}^{2d})$ be such that v and v_0 are submultiplicative, and ω_0 is v_0-moderate. Also let $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^{2d}$ with respect to v and $f \in \mathscr{S}'(\mathbf{R}^d)$. Then the following is true:*

(1) *if $\phi \in M^1_{(v_0 v)}(\mathbf{R}^d)\backslash 0$, then $f \in M(\omega, \mathscr{B})$ if and only if $V_\phi f \cdot \omega \in \mathscr{B}$. Furthermore, (1.7) defines a norm on $M(\omega, \mathscr{B})$, and different choices of ϕ give rise to equivalent norms;*

(2) $\mathscr{S}(\mathbf{R}^d) \subseteq M^1_{(v_0 v)}(\mathbf{R}^d) \subseteq M(\omega, \mathscr{B}) \subseteq M^\infty_{(1/(v_0 v))}(\mathbf{R}^d) \subseteq \mathscr{S}'(\mathbf{R}^d)$.

Proposition 1.6, (1) allows us to be rather vague about the choice of $\phi \in M^1_{(v)} \setminus 0$ in (1.7). For example, if $C > 0$ is a constant and Ω is a subset of $\mathscr{S}'$, then $\|a\|_{M(\omega,\mathscr{B})} \le C$ for every $a \in \Omega$, means that the inequality holds for some choice of $\phi \in M^1_{(v)} \setminus 0$ and every $a \in \Omega$. Evidently, for any other choice of $\phi \in M^1_{(v)} \setminus 0$, a similar inequality is true although C may have to be replaced by a larger constant, if necessary.

We recall that Fourier BF-spaces and modulation spaces are locally the same. In fact, let $\varphi \in \mathscr{S}(\mathbf{R}^d)\setminus 0$, $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^{2d}$, $\omega \in \mathscr{P}(\mathbf{R}^{2d})$ and set $\omega_0(\xi) = \omega(x_0,\xi)$ for some fixed $x_0 \in \mathbf{R}^d$. Then

$$\mathscr{B}_0 \equiv \{\, f \in \mathscr{S}'(\mathbf{R}^d)\, ;\, \varphi \otimes f \in \mathscr{B}\,\} \tag{1.8}$$

is a translation invariant BF-space on $\mathbf{R}^d$ under the norm $\|f\|_{\mathscr{B}_0} \equiv \|\varphi \otimes f\|_{\mathscr{B}}$. The space $\mathscr{B}_0$ is independent of $\varphi \in \mathscr{S}(\mathbf{R}^d)\setminus 0$, and different choices of φ gives rise of equivalent norms. Furthermore

$$M(\omega,\mathscr{B}) \cap \mathscr{E}'(\mathbf{R}^d) = \mathscr{F}\mathscr{B}_0(\omega_0) \cap \mathscr{E}'(\mathbf{R}^d) \tag{1.9}$$

(cf. [7]).

Next we recall some facts in Chapter XVIII in [31] concerning pseudo-differential operators. Let $a \in \mathscr{S}(\mathbf{R}^{2d})$, and $t \in \mathbf{R}$ be fixed. Then the pseudo-differential operator $\operatorname{Op}_t(a)$ is the linear and continuous operator on $\mathscr{S}(\mathbf{R}^d)$ defined by the formula

$$(\operatorname{Op}_t(a)f)(x) = (2\pi)^{-d} \iint a((1-t)x+ty,\xi)f(y)e^{i\langle x-y,\xi\rangle}\, dyd\xi. \tag{1.10}$$

For general $a \in \mathscr{S}'(\mathbf{R}^{2d})$, the pseudo-differential operator $\operatorname{Op}_t(a)$ is defined as the continuous operator from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$ with distribution kernel

$$K_{t,a}(x,y) = (2\pi)^{-d/2}(\mathscr{F}_2^{-1}a)((1-t)x+ty,x-y). \tag{1.11}$$

This definition makes sense, since $\mathscr{F}_2$ and the map

$$F \mapsto F\circ G_t \quad \text{with} \quad G_t(x,y) = ((1-t)x+ty,x-y),$$

are homeomorphisms on $\mathscr{S}'(\mathbf{R}^{2d})$. We also note that the latter definition of $\operatorname{Op}_t(a)$ agrees with the operator in (1.10) when $a \in \mathscr{S}(\mathbf{R}^{2d})$. If $t = 0$, then $\operatorname{Op}_t(a)$ is the Kohn-Nirenberg representation $\operatorname{Op}(a) = a(x,D)$.

If $a \in \mathscr{S}'(\mathbf{R}^{2d})$ and $s,t \in \mathbf{R}$, then there is a unique $b \in \mathscr{S}'(\mathbf{R}^{2d})$ such that $\operatorname{Op}_s(a) = \operatorname{Op}_t(b)$. By straight-forward applications of Fourier's inversion formula, it follows that

$$\operatorname{Op}_s(a) = \operatorname{Op}_t(b) \quad \iff \quad b(x,\xi) = e^{i(t-s)\langle D_x,D_\xi\rangle}a(x,\xi) \tag{1.12}$$

(cf. Section 18.5 in [31]).

Next we discuss our symbol classes. Let $m,\mu,r,\rho\in\mathbf{R}$ be fixed. Then $\mathrm{SG}^{m,\mu}_{r,\rho}(\mathbf{R}^{2d})$ is the set of all $a\in C^\infty(\mathbf{R}^{2d})$ such that for each pairs of multi-indices α and β, there is a constant $C_{\alpha,\beta}$ such that

$$|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)|\le C_{\alpha,\beta}\langle x\rangle^{m-r|\alpha|}\langle\xi\rangle^{\mu-\rho|\beta|}.$$

Usually we assume that $r,\rho\ge 0$ and that $\rho+r>0$.

More generally, assume that $\omega\in\mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$. Then we recall from the introduction that $\mathrm{SG}^{(\omega)}_{r,\rho}(\mathbf{R}^{2d})$ consists of all $a\in C^\infty(\mathbf{R}^{2d})$ such that for each pairs of multi-indices α and β, there are constants $C_{\alpha,\beta}$ such that (0.1) holds. We note that

$$\mathrm{SG}^{(\omega)}_{r,\rho}(\mathbf{R}^{2d})=S(\omega,g_{r,\rho}),\tag{1.13}$$

when $g=g_{r,\rho}$ is the Riemannian metric on $\mathbf{R}^{2d}$, defined by the formula

$$\big(g_{r,\rho}\big)_{(y,\eta)}(x,\xi)=\langle y\rangle^{-2r}|x|^2+\langle\eta\rangle^{-2\rho}|\xi|^2\tag{1.14}$$

(cf. Section 18.4–18.6 in [31]). Furthermore, $\mathrm{SG}^{(\omega)}_{r,\rho}=\mathrm{SG}^{m,\mu}_{r,\rho}$ when $\omega(x,\xi)=\langle x\rangle^m\langle\xi\rangle^\mu$.

The following result shows that pseudo-differential operators with symbols in $\mathrm{SG}^{(\omega)}_{r,\rho}$ behave well.

Proposition 1.7. *Let $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^{2d}$, $s,t\in\mathbf{R}$, $r,\rho\ge 0$, $\omega\in\mathscr{P}(\mathbf{R}^{2d})$, $\omega_0\in\mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ and that $a,b\in\mathscr{S}'(\mathbf{R}^{2d})$ are such that $\mathrm{Op}_s(a)=\mathrm{Op}_t(b)$. Then the following is true:*

1. *$a\in\mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$ if and only if $b\in\mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$;*
2. *if $a\in\mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$, then $\mathrm{Op}_t(a)$ is continuous on $\mathscr{S}(\mathbf{R}^d)$ and extends uniquely to a continuous operator on $\mathscr{S}'(\mathbf{R}^d)$;*
3. *if $a\in\mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$, then $\mathrm{Op}_t(a)$ is continuous from $M(\omega,\mathscr{B})$ to $M(\omega/\omega_0,\mathscr{B})$.*

Proof. From the assumptions it follows that $g_{r,\rho}$ in (1.14) is slowly varying, σ-temperate and satisfies $g_{r,\rho}\le g^\sigma_{r,\rho}$, and that ω is $g_{r,\rho}$-continuous and (σ, $g_{r,\rho}$)-temperate (see Sections 18.4–18.6 in [31] for definitions). The assertions (1) and (2) are now consequences of (1.13), Proposition 18.5.10 and Theorem 18.6.2 in [31].

Finally, (3) follows immediately from [53, Theorem 3.2]. The proof is complete. $\square$

If $a\in\mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$, then it follows from the definitions that there is a constant $C>0$ such that

$$|a(x,\xi)|\le C\omega_0(x,\xi).$$

On the other hand, a necessary and sufficient condition for a to be invertible, in the sense that $1/a$ should be a symbol in $\mathrm{SG}^{(1/\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$, is that for some constant $c>0$ we have

$$c\,\omega_0(x,\xi)\le|a(x,\xi)|.\tag{1.15}$$

A slightly relaxed condition is that (1.15) holds for some constant $c > 0$ and all points (x, ξ), outside a compact set $K \subseteq \mathbf{R}^{2d}$. In this case we say that a is *elliptic* (with respect to ω_0).

In the following we discuss more local invertibility conditions for symbols in $\mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$ in terms of the sets of characteristic points of the involved symbols. We remark that our definition of the set of characteristic points is slightly different comparing to [31, Definition 18.1.5] in view of Remark 1.17 below.

Definition 1.8. Assume that $r, \rho \ge 0$, $\omega_0 \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ and that $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$.

(1) a is called *ψ-invertible* with respect to ω_0 at the point $(x_0, \xi_0) \in \mathbf{R}^d \times (\mathbf{R}^d \backslash 0)$, if there exist a neighbourhood X of x_0, an open conical neighbourhood Γ of ξ_0 and positive constants R and c such that (1.15) holds for $x \in X$, $\xi \in \Gamma$ and $|\xi| \ge R$.
The point (x_0, ξ_0) is called *ψ-characteristic* for a with respect to ω_0 if a is *not* ψ-invertible with respect to ω_0 at (x_0, ξ_0);

(2) a is called *e-invertible* with respect to ω_0 at the point $(x_0, \xi_0) \in (\mathbf{R}^d \setminus 0) \times \mathbf{R}^d$, if there exist an open conical neighbourhood Γ of x_0, a neighbourhood X of ξ_0 and positive constants R and c such that (1.15) holds for $x \in \Gamma$, $|x| \ge R$ and $\xi \in X$.
The point (x_0, ξ_0) is called *e-characteristic* for a with respect to ω_0 if a is *not* e-invertible with respect to ω_0 at (x_0, ξ_0);

(3) a is called *ψe-invertible* with respect to ω_0 at the point $(x_0, \xi_0) \in (\mathbf{R}^d \setminus 0) \times (\mathbf{R}^d \setminus 0)$, if there exist open conical neighbourhoods Γ_1 of x_0 and Γ_2 of ξ_0, and positive constants R_1, R_2 and c such that (1.15) holds for $x \in \Gamma_1$, $|x| \ge R_1$, $\xi \in \Gamma_2$ and $|\xi| \ge R_2$.
The point (x_0, ξ_0) is called *ψe-characteristic* for a with respect to ω_0 if a is *not* ψe-invertible with respect to ω_0 at (x_0, ξ_0).

The set of characteristic points (the characteristic set), for a symbol $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$ with respect to ω_0, is denoted by

$$\mathrm{Char}(a) = \mathrm{Char}_{(\omega_0)}(a) = \mathrm{Char}^{\psi}_{(\omega_0)}(a) \cup \mathrm{Char}^{e}_{(\omega_0)}(a) \cup \mathrm{Char}^{\psi e}_{(\omega_0)}(a),$$

where the three components are the sets of points satisfying (1), (2) and (3), respectively.

Remark 1.9. Since the case $\omega_0 = 1$ in Definition 1.8 is especially important we also give the following definition. We say that $a \in \mathrm{SG}^{0,0}_{r,\rho}(\mathbf{R}^{2d})$ is *ψ-invertible* at $(x_0, \xi_0) \in \mathbf{R}^d \times (\mathbf{R}^d \backslash 0)$ if $(x_0, \xi_0) \notin \mathrm{Char}^{\psi}_{(\omega_0)}(a)$ with $\omega_0 = 1$. That is, there exist a neighbourhood X of x_0, an open conical neighbourhood Γ of ξ_0 and $R, c > 0$ such that (1.15) holds for $\omega_0 = 1$, $x \in X$ and $\xi \in \Gamma$ such that $|\xi| \ge R$.

In the same way, $a \in \mathrm{SG}^{0,0}_{r,\rho}(\mathbf{R}^{2d})$ is called *e-invertible* at $(x_0, \xi_0) \in (\mathbf{R}^d \backslash 0) \times \mathbf{R}^d$, if $(x_0, \xi_0) \notin \mathrm{Char}^{e}_{(\omega_0)}(a)$ with $\omega_0 = 1$, and a is called *ψe-invertible* at $(x_0, \xi_0) \in (\mathbf{R}^d \backslash 0) \times (\mathbf{R}^d \backslash 0)$, if $(x_0, \xi_0) \notin \mathrm{Char}^{\psi e}_{(\omega_0)}(a)$ with $\omega_0 = 1$.

In the next Definition we introduce different classes of cutoff functions, see also Definition 1.9 in [7].

Definition 1.10. Assume that $X \subseteq \mathbf{R}^d$ is open, $\Gamma \subseteq \mathbf{R}^d \setminus 0$ is an open cone, $x_0 \in X$ and that $\xi_0 \in \Gamma$.

(1) a smooth function φ on $\mathbf{R}^d$ is called a *cutoff function* with respect to x_0 and X, if $0 \le \varphi \le 1$, $\varphi \in C_0^\infty(X)$ and $\varphi = 1$ in an open neighbourhood of x_0. The set of cutoff functions with respect to x_0 and X is denoted by $\mathscr{C}_{x_0}(X)$;

(2) a smooth function ψ on $\mathbf{R}^d$ is called a *directional cutoff function* with respect to ξ_0 and Γ, if there is a constant $R > 0$ and open conical neighbourhood $\Gamma_1 \subseteq \Gamma$ of ξ_0 such that the following is true:
- $0 \le \psi \le 1$ and $\operatorname{supp} \psi \subseteq \Gamma$;
- $\psi(t\xi) = \psi(\xi)$ when $t \ge 1$ and $|\xi| \ge R$;
- $\psi(\xi) = 1$ when $\xi \in \Gamma_1$ and $|\xi| \ge R$.

The set of directional cutoff functions with respect to ξ_0 and Γ is denoted by $\mathscr{C}^{\mathrm{dir}}_{\xi_0}(\Gamma)$.

Remark 1.11. For notational convenience, the open neighbourhood X of x_0 and the open conical neighbourhood Γ of ξ_0 appearing in the previous definition will sometimes be omitted in the sequel, and we will simply write $\mathscr{C}_{x_0}$ and $\mathscr{C}^{\mathrm{dir}}_{\xi_0}$, respectively.

Remark 1.12. Let $x_1, \xi_1 \in \mathbf{R}^d$, $x_2, \xi_2 \in \mathbf{R}^d \backslash 0$, $\varphi_1 \in \mathscr{C}_{x_1}(\mathbf{R}^d)$, $\varphi_2 \in \mathscr{C}_{\xi_1}(\mathbf{R}^d)$, $\psi_1 \in \mathscr{C}^{\mathrm{dir}}_{x_2}(\mathbf{R}^d \backslash 0)$ and $\psi_2 \in \mathscr{C}^{\mathrm{dir}}_{\xi_2}(\mathbf{R}^d \backslash 0)$. Then

$$c_1 = \varphi_1 \otimes \psi_2, \quad c_2 = \psi_1 \otimes \varphi_2 \quad \text{and} \quad c_3 = \psi_1 \otimes \psi_2 \tag{1.16}$$

belong to $\mathrm{SG}^{0,0}_{1,1}(\mathbf{R}^{2d})$ and are ψ-invertible, e-invertible and ψe-invertible, respectively.

In the next three propositions we show that $\mathrm{Op}(a)$ satisfies convenient invertibility properties of the form

$$\mathrm{Op}(a)\,\mathrm{Op}(b) = \mathrm{Op}(c) + \mathrm{Op}(h), \tag{1.17}$$

outside the set of characteristic points for a symbol a. Here $\mathrm{Op}(b)$, $\mathrm{Op}(c)$ and $\mathrm{Op}(h)$ have the roles of "local inverse", "local identity" and smoothing operators respectively. From these propositions it also follows that our set of characteristic points in Definition 1.8 are related to those in [8, 31].

Proposition 1.13. *Let $r,\rho \in [0,1]$ be such that $\rho > 0$, $\omega_0 \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$, $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$, and let $(x_0,\xi_0) \in \mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$. Then the following conditions are equivalent:*

1. $(x_0,\xi_0) \notin \mathrm{Char}^{\psi}_{(\omega_0)}(a)$;
2. *there is an element $c \in \mathrm{SG}^{0,0}_{r,\rho}$ which is ψ-invertible at (x_0,ξ_0), and an element $b \in \mathrm{SG}^{(1/\omega_0)}_{r,\rho}$ such that $ab = c$;*
3. *there is an element $c \in \mathrm{SG}^{0,0}_{r,\rho}$ which is ψ-invertible at (x_0,ξ_0), and elements $h \in \mathrm{SG}^{0,-\rho}_{r,\rho}$ and $b \in \mathrm{SG}^{(1/\omega_0)}_{r,\rho}$ such that* (1.17) *holds;*
4. *for each neighbourhood X of x_0 and conical neighbourhood Γ of ξ_0, there is an element $c = \varphi \otimes \psi$ where $\varphi \in \mathscr{C}_{x_0}(X)$ and $\psi \in \mathscr{C}^{\mathrm{dir}}_{\xi_0}(\Gamma)$, and elements $h \in \mathscr{S}$ and $b \in \mathrm{SG}^{(1/\omega_0)}_{r,\rho}$ such that* (1.17) *holds. Furthermore, the supports of b and h are contained in $X \times \mathbf{R}^d$.*

Proposition 1.14. *Let $r,\rho \in [0,1]$ be such that $r > 0$, $\omega_0 \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$, $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$, and let $(x_0,\xi_0) \in (\mathbf{R}^d \setminus 0) \times \mathbf{R}^d$. Then the following conditions are equivalent:*

1. $(x_0,\xi_0) \notin \mathrm{Char}^{e}_{(\omega_0)}(a)$;
2. *there is an element $c \in \mathrm{SG}^{0,0}_{r,\rho}$ which is e-invertible at (x_0,ξ_0), and an element $b \in \mathrm{SG}^{(1/\omega_0)}_{r,\rho}$ such that $ab = c$;*
3. *there is an element $c \in \mathrm{SG}^{0,0}_{r,\rho}$ which is e-invertible at (x_0,ξ_0), and elements $h \in \mathrm{SG}^{-r,0}_{r,\rho}$ and $b \in \mathrm{SG}^{(1/\omega_0)}_{r,\rho}$ such that* (1.17) *holds;*
4. *for each open conical neighbourhood Γ of x_0 and open neighbourhood X of ξ_0, there is an element $c = \psi \otimes \varphi$ where $\psi \in \mathscr{C}^{\mathrm{dir}}_{x_0}(\Gamma)$ and $\varphi \in \mathscr{C}_{\xi_0}(X)$, and elements $h \in \mathscr{S}$ and $b \in \mathrm{SG}^{(1/\omega_0)}_{r,\rho}$ such that* (1.17) *holds. Furthermore, the supports of b and h are contained in $\Gamma \times \mathbf{R}^d$.*

Proposition 1.15. *Let $r,\rho \in (0,1]$, $\omega_0 \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$, $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$, and let $(x_0,\xi_0) \in (\mathbf{R}^d \setminus 0) \times (\mathbf{R}^d \setminus 0)$. Then the following conditions are equivalent:*

1. $(x_0,\xi_0) \notin \mathrm{Char}^{\psi e}_{(\omega_0)}(a)$;
2. *there is an element $c \in \mathrm{SG}^{0,0}_{r,\rho}$ which is ψe-invertible at (x_0,ξ_0), and an element $b \in \mathrm{SG}^{(1/\omega_0)}_{r,\rho}$ such that $ab = c$;*
3. *there is an element $c \in \mathrm{SG}^{0,0}_{r,\rho}$ which is ψe-invertible at (x_0,ξ_0), and elements $h \in \mathrm{SG}^{-r,0}_{r,\rho} + \mathrm{SG}^{0,-\rho}_{r,\rho}$ and $b \in \mathrm{SG}^{(1/\omega_0)}_{r,\rho}$ such that* (1.17) *holds;*
4. *for each open conical neighbourhoods Γ_1 of x_0 and Γ_2 of ξ_0, there is an element $c = \psi_1 \otimes \psi_2$ where $\psi_1 \in \mathscr{C}^{\mathrm{dir}}_{x_0}(\Gamma_1)$ and $\psi_2 \in$*

$\mathscr{C}^{\mathrm{dir}}_{\xi_0}(\Gamma_2)$, *and elements* $h \in \mathscr{S}$ *and* $b \in \mathrm{SG}^{(1/\omega_0)}_{r,\rho}$ *such that* (1.17) *holds. Furthermore, the supports of* b *and* h *are contained in* $\Gamma_1 \times \mathbf{R}^d$.

Propositions 1.13 and 1.14 are equivalent to each others, and they follow by the same arguments as in the proof of Proposition 2.3 in [7]. Proposition 1.15 follows by similar arguments. For completeness we give a proof of Proposition 1.15 in the Appendix.

As a consequence of Propositions 1.13–1.15, we can show that the sets of characteristic points are invariant under the choice of pseudo-differential calculus.

Proposition 1.16. *Assume that* $0 \le r,\rho \le 1$, $\omega_0 \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$, $s,t \in \mathbf{R}$ *and that* $a,b \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$ *satisfy*

$$\mathrm{Op}_s(a) = \mathrm{Op}_t(b).$$

Then the following is true:

(1) *if in addition* $\rho > 0$, *then* $\mathrm{Char}^{\psi}_{(\omega_0)}(a) = \mathrm{Char}^{\psi}_{(\omega_0)}(b)$;

(2) *if in addition* $r > 0$, *then* $\mathrm{Char}^{e}_{(\omega_0)}(a) = \mathrm{Char}^{e}_{(\omega_0)}(b)$;

(3) *if in addition* $r,\rho > 0$, *then* $\mathrm{Char}^{\psi e}_{(\omega_0)}(a) = \mathrm{Char}^{\psi e}_{(\omega_0)}(b)$.

For the proof and later references we set

$$\sigma_{m,\mu}(x,\xi) = \langle x\rangle^m \langle \xi\rangle^\mu .$$

when $m,\mu \in \mathbf{R}$.

Proof. We may assume that $s = 0$, and prove only (3). The other assertions follow by similar arguments and are left for the reader. By [31, Proposition 18.5.10], it follows that $b = a+h$, where $h \in \mathrm{SG}^{(\omega_0/\sigma_{r,\rho})}_{r,\rho}$. Then for each $\varepsilon > 0$ there is R such that $|h(x,\xi)| \le \varepsilon\omega_0(x,\xi)$ when $|x| \ge R$ or $|\xi| \ge R$. This implies that (3) in Definition 1.8 is fulfilled for a, if and only if it is fulfilled for b. This gives the result. □

Remark 1.17. Let $\omega(x,\xi) = \langle \xi\rangle^r$, $r \in \mathbf{R}$, and assume that $a \in \mathrm{SG}^{r,0}_{1,0}(\mathbf{R}^{2d})$ $= \mathrm{SG}^{(\omega)}_{1,0}(\mathbf{R}^{2d})$ is polyhomogeneous with principal symbol $a_r \in \mathrm{SG}^{r,0}_{1,0}(\mathbf{R}^{2d})$ (cf. Definition 18.1.5 in [31]). Also let $\mathrm{Char}'(a)$ be the set of characteristic points of $\mathrm{Op}(a)$ in the classical sense (i.e., in the sense of Definition 18.1.25 in [31]). Then

$$\mathrm{Char}^{\psi}_{(\omega)}(a) \subseteq \mathrm{Char}'(a), \tag{1.18}$$

where strict inclusion might appear (cf. Remark 1.4 and Example 3.11 in [38]).

2. Wave front sets with respect to Modulation spaces

In this section we define wave-front sets with respect to modulation spaces and show some of their properties. The basic ideas behind these definitions can be found in [8].

Definition 2.1. Let $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, $\mathscr{B}$ be a translation invariant BF-space and let $f \in \mathscr{S}'(\mathbf{R}^d)$.

(1) The wave-front set $\mathrm{WF}^{\psi}_{M(\omega,\mathscr{B})}(f)$ of *ψ-type* for f with respect to $M(\omega,\mathscr{B})$ consists of all $(x_0,\xi_0) \in \mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$ such that for every $\varphi \in \mathscr{C}_{x_0}(\mathbf{R}^d)$ and every $\psi \in \mathscr{C}^{\mathrm{dir}}_{\xi_0}(\mathbf{R}^d \setminus 0)$ it holds
$$\varphi \cdot \psi(D) f \notin M(\omega,\mathscr{B}); \tag{2.1}$$

(2) The wave-front set $\mathrm{WF}^{e}_{M(\omega,\mathscr{B})}(f)$ of *e-type* for f with respect to $M(\omega,\mathscr{B})$ consists of all $(x_0,\xi_0) \in (\mathbf{R}^d \setminus 0) \times \mathbf{R}^d$ such that for every $\psi \in \mathscr{C}^{\mathrm{dir}}_{x_0}(\mathbf{R}^d \setminus 0)$ and every $\varphi \in \mathscr{C}_{\xi_0}(\mathbf{R}^d)$ it holds
$$\psi \cdot \varphi(D) f \notin M(\omega,\mathscr{B}); \tag{2.2}$$

(3) The wave-front set $\mathrm{WF}^{\psi e}_{M(\omega,\mathscr{B})}(f)$ of *ψe-type* for f with respect to $M(\omega,\mathscr{B})$ consists of all $(x_0,\xi_0) \in (\mathbf{R}^d \setminus 0) \times (\mathbf{R}^d \setminus 0)$ such that for every $\psi_1 \in \mathscr{C}^{\mathrm{dir}}_{x_0}(\mathbf{R}^d \setminus 0)$ and every $\psi_2 \in \mathscr{C}^{\mathrm{dir}}_{\xi_0}(\mathbf{R}^d \setminus 0)$ it holds
$$\psi_1 \cdot \psi_2(D) f \notin M(\omega,\mathscr{B}). \tag{2.3}$$

Finally, the *global wave-front set* $\mathrm{WF}_{M(\omega,\mathscr{B})}(f) \subseteq (\mathbf{R}^d \times \mathbf{R}^d)\backslash 0$ is the set
$$\mathrm{WF}_{M(\omega,\mathscr{B})}(f) \equiv \mathrm{WF}^{\psi}_{M(\omega,\mathscr{B})}(f) \cup \mathrm{WF}^{e}_{M(\omega,\mathscr{B})}(f) \cup \mathrm{WF}^{\psi e}_{M(\omega,\mathscr{B})}(f).$$

Remark 2.2. In a similar way as in Remark 2.3 in [8], we note that Definition 2.1 does not change if the conditions
$$\varphi \cdot \psi(D) f \notin M(\omega,\mathscr{B}), \quad \psi \cdot \varphi(D) f \notin M(\omega,\mathscr{B}), \quad \text{and}$$
$$\psi_1 \cdot \psi_2(D) f \notin M(\omega,\mathscr{B})$$
in (2.1)–(2.3) are replaced by
$$\psi(D)(\varphi \cdot f) \notin M(\omega,\mathscr{B}), \quad \varphi(D)(\psi \cdot f) \notin M(\omega,\mathscr{B}), \quad \text{and}$$
$$\psi_2(D)(\psi_1 \cdot f) \notin M(\omega,\mathscr{B}),$$
respectively. In fact, let $c(x,\xi) = \varphi(x)\psi(\xi)$ where $\varphi \in \mathscr{C}_{x_0}(\mathbf{R}^d)$ and $\psi \in \mathscr{C}^{\mathrm{dir}}_{\xi_0}(\mathbf{R}^d \setminus 0)$, and let $c_1 \in \mathrm{SG}^{0,0}_{r,\rho}$ be equal to 1 on $\operatorname{supp} c$. Then it follows from the symbolic calculus that
$$\mathrm{Op}(c_1)\,\mathrm{Op}(c) = \mathrm{Op}(c)\,\mathrm{Op}(c_1) \bmod \mathrm{Op}(\mathscr{S}) = \mathrm{Op}(c) \bmod \mathrm{Op}(\mathscr{S}). \tag{2.4}$$
A combination of (2.4) and the fact that each pseudo-differential operator with symbol in $\mathrm{SG}^{0,0}_{r,\rho}$ is continuous on the modulation space $M(\omega,\mathscr{B})$ now shows that (1) in Definition 2.1 does not depend on the

order we apply operators φ and $\psi(D)$. The assertions for (2) and (3) follow in the same way.

Remark 2.3. Let $f \in \mathscr{S}'(\mathbf{R}^d)$, $x_0, \xi_0 \in \mathbf{R}^d \backslash 0$, $\psi_{j,1} \in \mathscr{C}_{x_0}^{\mathrm{dir}}(\mathbf{R}^d \backslash 0)$, $\psi_{j,2} \in \mathscr{C}_{\xi_0}^{\mathrm{dir}}(\mathbf{R}^d \backslash 0)$ for $j = 1, 2$ be such that $\psi_{1,k} = 1$ on $\operatorname{supp} \psi_{2,k}$ for $k = 1, 2$. Also let $\omega \in \mathscr{P}(\mathbf{R}^{2d})$ and $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^{2d}$. If $\psi_{1,1} \cdot \psi_{1,2}(D) f \in M(\omega, \mathscr{B})$, then $\psi_{2,1} \cdot \psi_{2,2}(D) f \in M(\omega, \mathscr{B})$.
In fact, if $c_j = \psi_{j,1} \otimes \psi_{j,2}$, then $c_1 = 1$ on $\operatorname{supp} c_2$, and it follows from the symbolic calculus that for some $h \in \mathscr{S}$ we have

$$\psi_{2,1} \cdot \psi_{2,2}(D) f = \operatorname{Op}(c_2) f = \operatorname{Op}(c_2) \operatorname{Op}(c_1) f + \operatorname{Op}(h) f.$$

The assertion now follows from the fact that $\operatorname{Op}(c_2)$ and $\operatorname{Op}(h)$ are continuous on $M(\omega, \mathscr{B})$, in view of Proposition 1.7.

We may also define wave-front sets with respect to Fourier BF-spaces, in a way similar to $\mathrm{WF}^{\psi}_{M(\omega,\mathscr{B})}(f)$, $\mathrm{WF}^{e}_{M(\omega,\mathscr{B})}(f)$ and $\mathrm{WF}^{\psi e}_{M(\omega,\mathscr{B})}(f)$ above. For example, if $\mathscr{B}$ is a translation invariant BF-space and $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, then the wave-front set $\mathrm{WF}^{\psi}_{\mathscr{F}\mathscr{B}(\omega)}(f)$ consists of all $(x_0, \xi_0) \in \mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$ such that for every $\varphi \in \mathscr{C}_{x_0}(\mathbf{R}^d)$ and every $\psi \in \mathscr{C}_{\xi_0}^{\mathrm{dir}}(\mathbf{R}^d \setminus 0)$ it holds

$$\varphi \cdot \psi(D) f \notin \mathscr{F}\mathscr{B}(\omega).$$

However, if $\mathscr{B}_0$ is defined by (1.8), then (1.9) gives

$$\mathrm{WF}^{\psi}_{M(\omega,\mathscr{B})}(f) = \mathrm{WF}^{\psi}_{\mathscr{F}\mathscr{B}_0(\omega)}(f). \tag{2.5}$$

The first type of wave-front sets with respect to general modulation space and Fourier BF-spaces were introduced in [7]. Here we recall these definitions and show that they agree with corresponding wave-front sets of ψ-type. Let $f \in \mathscr{S}'(\mathbf{R}^d)$, $\phi \in C_0^\infty(\mathbf{R}^d)$ and $\omega \in \mathscr{P}(\mathbf{R}^{2d})$. Also let χ_Γ be the characteristic function of Γ. Then $\mathrm{WF}'_{M(\omega,\mathscr{B})}(f)$ (denoted by $\mathrm{WF}_{M(\omega,\mathscr{B})}(f)$ in [7]) consists of all pairs $(x_0, \xi_0) \in \mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$ such that

$$\|(V_\phi(\varphi f)) \cdot (1 \otimes \chi_\Gamma) \cdot \omega\|_{\mathscr{B}} = +\infty$$

for every choice of open conical neighbourhood Γ of ξ_0 and $\varphi \in \mathscr{C}_{x_0}$. The wave-front set $\mathrm{WF}'_{\mathscr{F}\mathscr{B}(\omega)}(f)$ (denoted by $\mathrm{WF}_{\mathscr{F}\mathscr{B}(\omega)}(f)$ in [7]) consists of all pairs $(x_0, \xi_0) \in \mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$ such that $|\varphi f|_{\mathscr{F}\mathscr{B}(\omega,\Gamma)} = +\infty$ for every choice of open conical neighbourhood Γ of ξ_0 and $\varphi \in \mathscr{C}_{x_0}$. Here

$$|f|_{\mathscr{F}\mathscr{B}(\omega,\Gamma)} \equiv \|\widehat{f} \omega \chi_\Gamma\|_{\mathscr{B}}.$$

Proposition 2.4. *Let $f \in \mathscr{S}'(\mathbf{R}^d)$, $\mathscr{B}$ be a translation invariant BF-space, $\mathscr{B}_0$ be defined by* (1.8) *and let $\omega \in \mathscr{P}(\mathbf{R}^{2d})$. Then*

$$\mathrm{WF}^{\psi}_{M(\omega,\mathscr{B})}(f) = \mathrm{WF}'_{M(\omega,\mathscr{B})}(f) = \mathrm{WF}^{\psi}_{\mathscr{F}\mathscr{B}_0(\omega)}(f) = \mathrm{WF}'_{\mathscr{F}\mathscr{B}_0(\omega)}(f).$$

Proof. By Theorem 6.9 in [7] we have $\mathrm{WF}'_{M(\omega,\mathscr{B})}(f) = \mathrm{WF}'_{\mathscr{F}\mathscr{B}_0(\omega)}(f)$. Hence, in view of (2.5), it suffices to prove $\mathrm{WF}^{\psi}_{\mathscr{F}\mathscr{B}_0(\omega)}(f) = \mathrm{WF}'_{\mathscr{F}\mathscr{B}_0(\omega)}(f)$. By Remark 2.2 we have

$$
\begin{aligned}
& (x_0,\xi_0) \notin \mathrm{WF}'_{\mathscr{F}\mathscr{B}_0(\omega)}(f) \\
\iff \quad & |\varphi_{x_0} f|_{\mathscr{F}\mathscr{B}_0(\omega,\Gamma)} < \infty \qquad \text{for some } \varphi_{x_0} \in \mathscr{C}_{x_0} \text{ and } \Gamma = \Gamma_{\xi_0} \\
\iff \quad & \|\psi_{\xi_0}(D)(\varphi_{x_0} f)\|_{\mathscr{F}\mathscr{B}_0(\omega)} < \infty \quad \text{for some } \varphi_{x_0} \in \mathscr{C}_{x_0} \text{ and } \psi_{\xi_0} \in \mathscr{C}^{\mathrm{dir}}_{\xi_0} \\
\iff \quad & (x_0,\xi_0) \notin \mathrm{WF}^{\psi}_{\mathscr{F}\mathscr{B}_0(\omega)}(f).
\end{aligned}
$$

This proves the result. □

The next Proposition 2.5 gives an alternative definition of the global wave-front set in terms of intersection of the characteristic sets described in Section 1:

Proposition 2.5. *Let* $r,\rho \in [0,1]$, $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, $\omega_0 \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$, $f \in \mathscr{S}'(\mathbf{R}^d)$, *and let*

$$
\Omega = \{\, a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})\, ;\, \mathrm{Op}(a)f \in M(\omega/\omega_0,\mathscr{B})\,\}.
$$

Then

$$
\mathrm{WF}^{\psi}_{M(\omega,\mathscr{B})}(f) = \bigcap_{a\in\Omega} \mathrm{Char}^{\psi}_{(\omega_0)}(a), \quad \textit{when} \qquad \rho > 0 \tag{2.6}
$$

$$
\mathrm{WF}^{e}_{M(\omega,\mathscr{B})}(f) = \bigcap_{a\in\Omega} \mathrm{Char}^{e}_{(\omega_0)}(a), \quad \textit{when} \qquad r > 0 \tag{2.7}
$$

$$
\mathrm{WF}^{\psi e}_{M(\omega,\mathscr{B})}(f) = \bigcap_{a\in\Omega} \mathrm{Char}^{\psi e}_{(\omega_0)}(a) \quad \textit{when} \quad r,\rho > 0. \tag{2.8}
$$

Proof. Let $b \in \mathrm{SG}^{(1/\omega_0)}_{r,\rho}$ and $b_1 \in \mathrm{SG}^{(\omega_0)}_{r,\rho}$ be such that

$$
\mathrm{Op}(b_1)\,\mathrm{Op}(b) = \mathrm{Op}(b)\,\mathrm{Op}(b_1) \tag{2.9}
$$

is the identity operator on L^2. It is possible to find such operators, in view of [3, Corollary 7.5]. By Theorem 2.1 in [27] it follows that the operator in (2.9) is the identity operator on each modulation space. Since

$$
\mathrm{Op}(b)\,\mathrm{Op}(a) \in \mathrm{Op}(\mathrm{SG}^{(1)}_{r,\rho}) = \mathrm{Op}(\mathrm{SG}^{0,0}_{r,\rho})
$$

when $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}$, we may assume that $\omega_0 = 1$.

In order to prove (2.6), we first assume that $(x_0,\xi_0) \notin \mathrm{WF}^{\psi}_{M(\omega,\mathscr{B})}(f)$. By Definition 2.1 there exists $\varphi \in \mathscr{C}_{x_0}$ and $\psi \in \mathscr{C}^{\mathrm{dir}}_{\xi_0}$ such that $a(x,\xi) \equiv \varphi(x)\psi(\xi) \in \mathrm{SG}^{0,0}_{1,0}$ and $\mathrm{Op}(a)f \in M(\omega,\mathscr{B})$. Since (1.15) is fulfilled with $\omega = 1$, for some constant $c > 0$, it follows that a is ψ-invertible at (x_0,ξ_0). Hence $(x_0,\xi_0) \notin \mathrm{Char}^{\psi}_{(\omega_0)}(a)$, and we have proved that $\cap\,\mathrm{Char}^{\psi}_{(\omega_0)}(a) \subseteq \mathrm{WF}^{\psi}_{M(\omega,\mathscr{B})}(f)$.

It remains to prove the opposite inclusion. Let $a \in \mathrm{SG}^{0,0}_{r,\rho}$ be such that $(x_0,\xi_0) \notin \mathrm{Char}^{\psi}(a)$ and $\mathrm{Op}(a)f \in M(\omega,\mathscr{B})$. By Proposition 1.13, there are $\varphi \in \mathscr{C}_{x_0}$, $\psi \in \mathscr{C}^{\mathrm{dir}}_{\xi_0}$, $b \in \mathrm{SG}^{0,0}_{r,\rho}$ and $h \in \mathscr{S}$ such that

$$\mathrm{Op}(\varphi \otimes \psi) = \mathrm{Op}(b)\,\mathrm{Op}(a) + \mathrm{Op}(h).$$

Since $\mathrm{Op}(a)f \in M(\omega,\mathscr{B})$, $\mathrm{Op}(b)$ is continuous on $M(\omega,\mathscr{B})$, and $\mathrm{Op}(h)$ maps $\mathscr{S}'$ into $\mathscr{S}$, it follows that $\varphi\cdot(\psi(D)f) = \mathrm{Op}(\varphi\otimes\psi)f \in M(\omega,\mathscr{B})$. Hence $(x_0,\xi_0) \notin \mathrm{WF}^{\psi}_{M(\omega,\mathscr{B})}(f)$. This proves (2.6). By similar arguments we also get (2.7) and (2.8). The details are left for the reader, and the proof is complete. $\square$

The next result describes the relation between "regularity in modulation spaces" of temperate distributions and global wave-front sets:

Theorem 2.6. *Let $\omega \in \mathscr{P}(\mathbf{R}^{2d})$ and $f \in \mathscr{S}'(\mathbf{R}^d)$. Then*

$$f \in M(\omega,\mathscr{B}) \quad \Longleftrightarrow \quad \mathrm{WF}_{M(\omega,\mathscr{B})}(f) = \emptyset.$$

For the proof we need the following lemma:

Lemma 2.7. *Let $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, $f \in \mathscr{S}'(\mathbf{R}^d)$ and let $\mathscr{B}$ be a translation invariant BF-space. Then the following is true:*

(1) *if $\mathrm{WF}^{\psi}_{M(\omega,\mathscr{B})}(f) = \emptyset$, then for each bounded open set $X \subseteq \mathbf{R}^d$, there exists a non-negative $a \in \mathrm{SG}^{0,0}_{1,1}$ such that $a \ge 1$ on $X \times \mathbf{R}^d$ and $\mathrm{Op}(a)f \in M(\omega,\mathscr{B})$;*

(2) *if $\mathrm{WF}^{e}_{M(\omega,\mathscr{B})}(f) = \emptyset$, then for each bounded open set $X \subseteq \mathbf{R}^d$, there exists a non-negative $a \in \mathrm{SG}^{0,0}_{1,1}$ such that $a \ge 1$ on $\mathbf{R}^d \times X$ and $\mathrm{Op}(a)f \in M(\omega,\mathscr{B})$;*

(3) *if $\mathrm{WF}^{\psi e}_{M(\omega,\mathscr{B})}(f) = \emptyset$, then for some bounded open sets $X_1, X_2 \subseteq \mathbf{R}^d$ such that $0 \in X_1$ and $0 \in X_2$, there exists a non-negative $a \in \mathrm{SG}^{0,0}_{1,1}$ such that $a \ge 1$ on $(\mathbf{R}^d \setminus X_1) \times (\mathbf{R}^d \setminus X_2)$ and $\mathrm{Op}(a)f \in M(\omega,\mathscr{B})$.*

Proof. We only prove (1). The other assertions follow by similar arguments and are left for the reader.

The condition $\mathrm{WF}^{\psi}_{M(\omega,\mathscr{B})}(f) = \emptyset$ implies that for each $(x_0,\xi_0) \in \mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$, there are functions $\varphi_{x_0,1} \in \mathscr{C}_{x_0}$ and $\psi_{\xi_0,1} \in \mathscr{C}^{\mathrm{dir}}_{\xi_0}$ such that $\varphi_{x_0,1}\cdot\psi_{\xi_0,1}(D)f \in M(\omega,\mathscr{B})$. Now we recall that each closed cone in $\mathbf{R}^d \setminus 0$ corresponds to a compact set on the unit sphere. Hence, by compactness, it follows that for some $\varphi_{x_0,2} \in \mathscr{C}_{x_0}$, $\xi_1,\dots,\xi_N$ and some constants $c, R > 0$, we have

$$\psi_1(\xi) = \sum_{j=1}^{N} \psi_{\xi_j,1}(\xi) > c, \quad \text{when} \quad |\xi| \ge R,$$

$\varphi_{x_0,2} \otimes \psi_1 \in \mathrm{SG}^{0,0}_{1,1}$ and $\varphi_{x_0,2}\cdot\psi_1(D)f \in M(\omega,\mathscr{B})$.

Now choose non-negative $\varphi_3 \in C_0^\infty(\mathbf{R}^d)$ such that $\varphi_3(\xi) = 1$ when $|\xi| \le R$. Then $\varphi_{x_0,2} \cdot \varphi_3(D)f \in C_0^\infty(\mathbf{R}^d) \subseteq M(\omega, \mathscr{B})$, since $\varphi_{x_0,2} \otimes \varphi_3 \in C_0^\infty(\mathbf{R}^{2d})$. Hence, for some $\varphi_{x_0} \in \mathscr{C}_{x_0}$, open neighbourhood $U = U_{x_0}$ of x_0 and some constant $C > 0$, the element $a_{x_0} = C\varphi_{x_0} \otimes (\psi_1 + \varphi_3)$ belongs to $\mathrm{SG}^{0,0}_{1,1}$ and is larger than 1 on $U \times \mathbf{R}^d$. Furthermore, $\mathrm{Op}(a_{x_0})f \in M(\omega, \mathscr{B})$.

For each compact set K we may find finite numbers of $U_{x_1}, \dots U_{x_N}$ which cover K. The result now follows if we choose

$$a = a_{x_1} + \cdots + a_{x_N}.$$

□

Proof of Theorem 2.6. The right implication is obvious by Definition 2.1, since operators in $\mathrm{Op}(\mathrm{SG}^{0,0}_{r,\rho})$ are continuous on $M(\omega, \mathscr{B})$.

Assume that $\mathrm{WF}_{M(\omega,\mathscr{B})}(f) = \emptyset$. Then $\mathrm{WF}^{\psi}_{M(\omega,\mathscr{B})}(f) = \mathrm{WF}^{e}_{M(\omega,\mathscr{B})}(f) = \mathrm{WF}^{\psi e}_{M(\omega,\mathscr{B})}(f) = \emptyset$. By Lemma 2.7 (3), there is a non-negative element $a_3 \in \mathrm{SG}^{0,0}_{1,1}$, open sets X_1, X_2 such that $0 \in X_1$, $0 \in X_2$, $a_3 \ge 1$ in $(\mathbf{R}^d \setminus X_1) \times (\mathbf{R}^d \setminus X_2)$ and $\mathrm{Op}(a_3)f \in M(\omega, \mathscr{B})$. Furthermore, by Lemma 2.7 (1) and (2), there are non-negative elements $a_1, a_2 \in \mathrm{SG}^{0,0}_{1,1}$ such that $a_1 \ge 1$ in $X_1 \times \mathbf{R}^d$, $a_2 \ge 1$ in $\mathbf{R}^d \times X_2$, $\mathrm{Op}(a_1)f \in M(\omega, \mathscr{B})$ and $\mathrm{Op}(a_2)f \in M(\omega, \mathscr{B})$. Hence, if $a = a_1 + a_2 + a_3$, it follows that $a \in \mathrm{SG}^{0,0}_{1,1}$, $\mathrm{Op}(a)f \in M(\omega, \mathscr{B})$, and that $a \ge 1$. Hence a is elliptic in $\mathrm{SG}^{0,0}_{1,1}$. The ellipticity of a implies that for some $b \in \mathrm{SG}^{0,0}_{1,1}$ and $h \in \mathrm{SG}^{-\infty,-\infty}_{1,1} = \mathscr{S}$ we have

$$\mathrm{Op}(b)\,\mathrm{Op}(a) = \mathrm{Id} + \mathrm{Op}(h)$$

(cf. the proof of Proposition 1.15). Since $\mathrm{Op}(b)$ and $\mathrm{Op}(h)$ are continuous on $M(\omega, \mathscr{B})$ we get

$$f = \mathrm{Op}(b)\,\mathrm{Op}(a)f - \mathrm{Op}(h)f \in M(\omega, \mathscr{B}),$$

and the assertion follows. The proof is complete. □

We end the section by giving some remarks on mapping properties for wave-front sets under Fourier transform. Here it is convenient to let ω_T be the composition of the weight $\omega \in \mathscr{P}(\mathbf{R}^{2d})$ with the torsion $T(x, \xi) = (-\xi, x)$, and $\mathscr{B}_T$ denote the space of the pull-backs of the elements of the translation invariant BF-space $\mathscr{B}$ on $\mathbf{R}^{2d}$ under T. That is,

$$\mathscr{B}_T = \{\, F \circ T \,;\, F \in \mathscr{B} \,\}, \quad \text{and} \qquad \omega_T = \omega \circ T \\ \text{where} \quad T(x,\xi) = (-\xi, x). \tag{2.10}$$

The first two equalities in the following proposition are related to Lemma 2.4 in [8].

Proposition 2.8. *Let $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^{2d}$, $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, and let T, $\mathscr{B}_T$ and ω_T be as in* (2.10)*. If $f \in \mathscr{S}'(\mathbf{R}^{2d})$, then*

$$T(\mathrm{WF}^{\psi}_{M(\omega,\mathscr{B})}(f)) = \mathrm{WF}^{e}_{M(\omega_T,\mathscr{B}_T)}(\widehat{f}),$$

$$T(\mathrm{WF}^{e}_{M(\omega,\mathscr{B})}(f)) = \mathrm{WF}^{\psi}_{M(\omega_T,\mathscr{B}_T)}(\widehat{f}),$$

$$T(\mathrm{WF}^{\psi e}_{M(\omega,\mathscr{B})}(f)) = \mathrm{WF}^{\psi e}_{M(\omega_T,\mathscr{B}_T)}(\widehat{f}).$$

Proof. By Fourier's inversion formula we have

$$|(V_\phi f)\circ T| = |V_{\widehat{\phi}}\widehat{f}|, \quad \mathscr{F}(a\cdot(b(D)f)) = \check{a}(D)(b\cdot\widehat{f}).$$

The result is now a straight-forward consequence of these identities, Remark 2.2 and the definitions. The details are left for the reader. □

3. Wave-front sets for pseudo-differential operators with smooth symbols

In this section we consider mapping properties for pseudo-differential operators with respect to our global wave-front sets. More precisely, we prove that microlocality and microellipticity hold for pseudodifferential operators in $\operatorname{Op}(\mathrm{SG}^{(\omega_0)}_{r,\rho})$. For notational convenience, we set

$$\mathcal{B} = M(\omega,\mathscr{B}) \quad \text{and} \quad \mathcal{C} = M(\omega/\omega_0,\mathscr{B}). \tag{3.1}$$

We start with the following result:

Theorem 3.1. *Let $r,\rho \in [0,1]$, $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^{2d}$, $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, $\omega_0 \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$, $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^d)$ and let $f \in \mathscr{S}'(\mathbf{R}^d)$. Moreover, let $\mathcal{B}$ and $\mathcal{C}$ be defined as in* (3.1)*. Then the following is true:*

(1) *if in addition $\rho > 0$, then*

$$\mathrm{WF}^{\psi}_{\mathcal{C}}(\operatorname{Op}(a)f) \subseteq \mathrm{WF}^{\psi}_{\mathcal{B}}(f) \subseteq \mathrm{WF}^{\psi}_{\mathcal{C}}(\operatorname{Op}(a)f) \cup \mathrm{Char}^{\psi}_{(\omega_0)}(a); \tag{3.2}$$

(2) *if in addition $r > 0$, then*

$$\mathrm{WF}^{e}_{\mathcal{C}}(\operatorname{Op}(a)f) \subseteq \mathrm{WF}^{e}_{\mathcal{B}}(f) \subseteq \mathrm{WF}^{e}_{\mathcal{C}}(\operatorname{Op}(a)f) \cup \mathrm{Char}^{e}_{(\omega_0)}(a); \tag{3.3}$$

(3) *if in addition $r,\rho > 0$, then*

$$\mathrm{WF}^{\psi e}_{\mathcal{C}}(\operatorname{Op}(a)f) \subseteq \mathrm{WF}^{\psi e}_{\mathcal{B}}(f) \subseteq \mathrm{WF}^{\psi e}_{\mathcal{C}}(\operatorname{Op}(a)f) \cup \mathrm{Char}^{\psi e}_{(\omega_0)}(a). \tag{3.4}$$

Proof. The assertion (1) follows from Theorem 4.1 in [7] and Proposition 2.4, and the assertion (2) follows by similar arguments. We have to prove (3.4), and we begin by proving the first inclusion.

Assume that $(x_0,\xi_0) \notin \mathrm{WF}^{\psi e}_{M(\omega,\mathscr{B})}(f)$. We shall prove that $(x_0,\xi_0) \notin \mathrm{WF}^{\psi e}_{M(\omega/\omega_0,\mathscr{B})}(\operatorname{Op}(a)f)$. For some $\psi_{1,1} \in \mathscr{C}^{\mathrm{dir}}_{x_0}$ and $\psi_{1,2} \in \mathscr{C}^{\mathrm{dir}}_{\xi_0}$ we have

$$\psi_{1,1}\cdot\psi_{1,2}(D)f \in M(\omega,\mathscr{B}), \tag{3.5}$$

in view of (3) in Definition 2.1. We choose $\psi_{2,1} \in \mathscr{C}_{x_0}^{\mathrm{dir}}$ and $\psi_{2,2} \in \mathscr{C}_{\xi_0}^{\mathrm{dir}}$ such that $\psi_{1,j} = 1$ on $\operatorname{supp}\psi_{2,j}$, and we set

$$c_1(x,\xi) = \psi_{1,1}(x)\psi_{1,2}(\xi) \quad \text{and} \quad c_2(x,\xi) = \psi_{2,1}(x)\psi_{2,2}(\xi).$$

Then $c_1, c_2 \in \mathrm{SG}^{0,0}_{r,\rho}$, and since $c_1 = 1$ on $\operatorname{supp} c_2$ and that $\mathrm{SG}^{-\infty,-\infty}_{r,\rho} = \mathscr{S}$, it follows from the symbolic calculus that

$$\operatorname{Op}(c_2)\operatorname{Op}(a) = \operatorname{Op}(c_2)\operatorname{Op}(a)\operatorname{Op}(c_1) \mod \operatorname{Op}(\mathscr{S}). \tag{3.6}$$

Now we recall that the mappings

$$\begin{aligned} \operatorname{Op}(a) \,:\, M(\omega,\mathscr{B}) &\to M(\omega/\omega_0,\mathscr{B}), \\ \operatorname{Op}(c_2) \,:\, M(\omega/\omega_0,\mathscr{B}) &\to M(\omega/\omega_0,\mathscr{B}) \end{aligned} \tag{3.7}$$

are continuous (cf. Proposition 1.7). The combination of (3.5)–(3.7), of the fact that $\operatorname{Op}(c_1) = \psi_{1,1}\psi_{1,2}(D)$, and that $\operatorname{Op}(h)$ maps $\mathscr{S}'$ into $\mathscr{S}$ give

$$\begin{aligned} \psi_{2,1}\cdot\psi_{2,2}(D)(\operatorname{Op}(a)f) &= \operatorname{Op}(c_2)\operatorname{Op}(a)f \\ &= \operatorname{Op}(c_2)\operatorname{Op}(a)\operatorname{Op}(c_1)f \mod \mathscr{S} \in M(\omega/\omega_0,\mathscr{B}). \end{aligned}$$

This proves that $(x_0,\xi_0) \notin \mathrm{WF}^{\psi e}_{\mathcal{C}}(\operatorname{Op}(a)f)$, and the first inclusion in (3.4) follows.

It remains to prove the second inclusion in (3.4). Assume that

$$(x_0,\xi_0) \notin \mathrm{WF}^{\psi e}_{\mathcal{C}}(\operatorname{Op}(a)f) \cup \operatorname{Char}^{\psi e}_{(\omega_0)}(a).$$

By Remark 2.3, there exist $\psi_{1,1} \in \mathscr{C}_{x_0}^{\mathrm{dir}}$ and $\psi_{1,2} \in \mathscr{C}_{\xi_0}^{\mathrm{dir}}$, $b \in \mathrm{SG}^{(1/\omega_0)}_{r,\rho}$ and $h \in \mathscr{S}$ such that

$$\psi_{1,1}\cdot\psi_{1,2}(D)(\operatorname{Op}(a)f) \in M(\omega/\omega_0,\mathscr{B})$$

and (1.17) holds for $c = c_1 \equiv \psi_{1,1}\otimes\psi_{1,2}$. We claim that

$$\operatorname{Op}(c_2) = \operatorname{Op}(c_2)\operatorname{Op}(b)\operatorname{Op}(c_1)\operatorname{Op}(a) + \operatorname{Op}(h), \tag{3.8}$$

for some $h \in \mathscr{S}$, where $c_2 = \psi_{2,1}\otimes\psi_{2,2}$, and $\psi_{2,1} \in \mathscr{C}_{x_0}^{\mathrm{dir}}$ and $\psi_{2,2} \in \mathscr{C}_{\xi_0}^{\mathrm{dir}}$ are such that $\psi_{1,j} = 1$ on $\operatorname{supp}\psi_{2,j}$ for $j = 1,2$.

In fact, by composing (1.17) with $\operatorname{Op}(c_2)$ and using the fact that $\operatorname{Op}(c_2)\operatorname{Op}(c_1) = \operatorname{Op}(c_2) \mod \operatorname{Op}(\mathscr{S})$, we get

$$\begin{aligned} \operatorname{Op}(c_2) = \operatorname{Op}(c_2)\operatorname{Op}(b)\operatorname{Op}(a) &\mod \operatorname{Op}(\mathscr{S}) \\ = \operatorname{Op}(c_2)\operatorname{Op}(c_1)\operatorname{Op}(b)\operatorname{Op}(a) &\mod \operatorname{Op}(\mathscr{S}) \\ = \operatorname{Op}(c_2)\operatorname{Op}(b)\operatorname{Op}(c_1)\operatorname{Op}(a) &\mod \operatorname{Op}(\mathscr{S}), \end{aligned}$$

and (3.8) follows. Here the last equality follows from the fact that

$$\operatorname{Op}(c_2)[\operatorname{Op}(b),\operatorname{Op}(c_1)] \in \operatorname{Op}(\mathscr{S}),$$

when $c_1 = 1$ on $\operatorname{supp} c_2$, where $[\,\cdot\,,\,\cdot\,]$ denotes the commutator. The combination of Proposition 1.7 and (3.8) with the fact that $\operatorname{Op}(c_1)(\operatorname{Op}(a)f)$

$\in M(\omega/\omega_0,\mathscr{B})$ now shows that the mappings

$$\begin{aligned}\operatorname{Op}(b)\ &: M(\omega/\omega_0,\mathscr{B}) \to M(\omega,\mathscr{B})\\ \operatorname{Op}(c_2)\ &: M(\omega,\mathscr{B}) \qquad \to M(\omega,\mathscr{B})\end{aligned}$$

and

$$\operatorname{Op}(h)\ :\ \mathscr{S}' \ \to\ \mathscr{S}$$

are continuous and that $\operatorname{Op}(c_2)f \in M(\omega,\mathscr{B})$. Hence, we have showed that $(x_0,\xi_0)\notin \mathrm{WF}^{\psi e}_{\mathcal{B}}(f)$, and the proof is complete. □

Corollary 3.2. *Let $r>0$, $f\in\mathscr{S}'(\mathbf{R}^d)$ and $\varphi\in C^\infty(\mathbf{R}^d)$ be such that $\langle x\rangle^{r|\alpha|}\partial^\alpha\varphi(x)\in L^\infty(\mathbf{R}^d)$ for every α. Also let $\mathcal{B}=M(\omega,\mathscr{B})$, where $\mathscr{B}$ is an invariant BF-space on $\mathbf{R}^{2d}$ and $\omega\in\mathscr{P}(\mathbf{R}^{2d})$. Then*

$$\begin{gathered}\mathrm{WF}^{\psi}_{\mathcal{B}}(\varphi f)\subseteq \mathrm{WF}^{\psi}_{\mathcal{B}}(f),\ \mathrm{WF}^{e}_{\mathcal{B}}(\varphi f)\subseteq \mathrm{WF}^{e}_{\mathcal{B}}(f)\\ \text{and}\quad \mathrm{WF}^{\psi e}_{\mathcal{B}}(\varphi f)\subseteq \mathrm{WF}^{\psi e}_{\mathcal{B}}(f).\end{gathered}\tag{3.9}$$

Proof. It follows from the assumptions that $a=\varphi\otimes 1\in SG^{0,0}_{r,1}$. Hence Theorem 3.1 gives

$$\mathrm{WF}^{\psi e}_{\mathcal{B}}(\varphi f)=\mathrm{WF}^{\psi e}_{\mathcal{B}}(\operatorname{Op}(a)f)\subseteq \mathrm{WF}^{\psi e}_{\mathcal{B}}(f),$$

which is the last inclusion in (3.9): the other inclusions follow by analogous arguments. The proof is complete. □

Next we apply Theorem 3.1 on operators which are elliptic with respect to $\omega_0\in\mathscr{P}_{\rho,\delta}(\mathbf{R}^{2d})$ when $0<r,\rho\le 1$. We recall that a and $\operatorname{Op}(a)$ are called SG-*elliptic* with respect to $\mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$ or ω_0, if there is a compact set $K\subset\mathbf{R}^{2d}$ and a positive constant c such that (1.15) holds when $(x,\xi)\notin K$. Since $|a(x,\xi)|\le C\omega_0(x,\xi)$, it follows from the definitions that for each multi-index α, there are constants $C_{\alpha,\beta}$ such that

$$|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)|\le C_{\alpha,\beta}|a(x,\xi)|\langle x\rangle^{-r|\alpha|}\langle\xi\rangle^{-\rho|\beta|},\qquad (x,\xi)\in\mathbf{R}^{2d}\setminus K,$$

when a is SG-elliptic (see, e.g., [2,31]).

It follows from Lemma 2.7 that $\mathrm{Char}_{(\omega_0)}(a)=\emptyset$ if and only if a is SG-elliptic with respect to ω_0. The following result is now an immediate consequence of Theorem 3.1:

Theorem 3.3. *Let $r,\rho\in(0,1]$, $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^{2d}$, $\omega\in\mathscr{P}(\mathbf{R}^{2d})$, $\omega_0\in\mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$, $a\in\mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^d)$ be SG-elliptic with respect to ω_0 and let $f\in\mathscr{S}'(\mathbf{R}^d)$. Moreover, let $\mathcal{B}$ and $\mathcal{C}$ be defined as in (3.1). Then*

$$\begin{aligned}\mathrm{WF}^{\psi}_{\mathcal{C}}(\operatorname{Op}(a)f)&=\mathrm{WF}^{\psi}_{\mathcal{B}}(f),\\ \mathrm{WF}^{e}_{\mathcal{C}}(\operatorname{Op}(a)f)&=\mathrm{WF}^{e}_{\mathcal{B}}(f),\\ \mathrm{WF}^{\psi e}_{\mathcal{C}}(\operatorname{Op}(a)f)&=\mathrm{WF}^{\psi e}_{\mathcal{B}}(f).\end{aligned}$$

Theorem 3.4. *Assume that the hypotheses in Theorem 3.3 are fulfilled, let $g \in M(\omega/\omega_0, \mathscr{B})$ and let $f \in \mathscr{S}'(\mathbf{R}^d)$ be a solution to the equation*

$$\operatorname{Op}(a)f = g. \tag{3.10}$$

Then, $f \in M(\omega, \mathscr{B})$.

Proof. The result follows by combining Theorems 2.6 and 3.3. □

We remark that the assumptions $r > 0$, $\rho > 0$ or $r, \rho > 0$, respectively, are necessary in order to show that the conclusions in Theorem 3.1 hold true (see, e. g., Remark 3.7 in [38]).

Example 3.5. Let $0 < r, \rho \le 1$, $a(x,\xi) = a_1(x,\xi) + a_2(\xi)$ be such that the following conditions are fulfilled:

(1) $|a(x,\xi)| \ge c$ for some constant $c > 0$;
(2) $a_1 \in \mathrm{SG}^{0,0}_{r,\rho}(\mathbf{R}^{2d})$;
(3) a_2 is the symbol of a linear partial differential operator with constant coefficient which is hypoelliptic in the sense of [31].

Then a is elliptic with respect to

$$\omega_0(x,\xi) = (1 + |a_2(\xi)|^2)^{1/2},$$

which belongs to $\mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$. Hence we may apply Theorems 3.3 and 3.4 on $\operatorname{Op}(a)$.

An interesting case concerns the modified heat operator $a_1(x,t) + \partial_t - \Delta_x$, where $(x,t) \in \mathbf{R}^{d+1}$, $a_1(x,t) \in C^\infty(\mathbf{R}^{d+1})$ and a_1 is equal to $c > 0$ outside a compact set in $\mathbf{R}^{d+1}$. The symbol of the operator is $a(x,t,\xi,\tau) = a_1(x,t) + |\xi|^2 + i\tau$. In this case, a is elliptic with respect to

$$\omega_0(x,t,\xi,\tau) = (1 + |\xi|^4 + |\tau|^2)^{1/2}.$$

Hence, if $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, $\mathscr{B}$ is a translation invariant BF-space on $\mathbf{R}^d$, and (3.10) holds for some $f \in \mathscr{S}'(\mathbf{R}^d)$ and $g \in M(\omega/\omega_0, \mathscr{B})$, then it follows from Theorem 3.4 that $f \in M(\omega, \mathscr{B})$.

4. Wave-front sets of sup and inf types

In this section we define wave-front sets based on sequences of modulation spaces, and discuss basic results. More precisely, we consider wave-front sets with respect to sequences of the form

$$(\mathcal{B}_j) \equiv (\mathcal{B}_j)_{j\in J}, \quad \text{with} \quad \mathcal{B}_j = M(\omega_j, \mathscr{B}_j), \tag{4.1}$$

where $\omega_j \in \mathscr{P}(\mathbf{R}^{2d})$, $\mathscr{B}_j$ is a translation invariant BF-spaces on $\mathbf{R}^d$, and j belongs to some index set J.

Definition 4.1. Let J be an index set, $\mathscr{B}_j$ be translation invariant BF-spaces on $\mathbf{R}^d$, $\omega_j \in \mathscr{P}(\mathbf{R}^{2d})$ when $j \in J$, $(\mathcal{B}_j)$ be as in (4.1), and let $f \in \mathscr{S}'(\mathbf{R}^d)$.

(1) The wave-front set $\mathrm{WF}^{\psi,\sup}_{(\mathcal{B}_j)}(f)$ of sup-type ($\mathrm{WF}^{\psi,\inf}_{(\mathcal{B}_j)}(f)$ of inf-type) with respect to $(\mathcal{B}_j)$, consists of all pairs (x_0,ξ_0) in $\mathbf{R}^d\times(\mathbf{R}^d\backslash 0)$ such that for every $\varphi\in\mathscr{C}_{x_0}(\mathbf{R}^d)$ and every $\psi\in\mathscr{C}^{\mathrm{dir}}_{\xi_0}(\mathbf{R}^d\backslash 0)$ it holds

$$\varphi\cdot\psi(D)f\notin M(\omega_j,\mathscr{B}_j) \tag{4.2}$$

for some $j\in J$ (for every $j\in J$);

(2) The wave-front set $\mathrm{WF}^{e,\sup}_{(\mathcal{B}_j)}(f)$ of sup-type ($\mathrm{WF}^{e,\inf}_{(\mathcal{B}_j)}(f)$ of inf-type) with respect to $(\mathcal{B}_j)$, consists of all pairs (x_0,ξ_0) in $(\mathbf{R}^d\setminus 0)\times\mathbf{R}^d$ such that for every $\varphi\in\mathscr{C}^{\mathrm{dir}}_{x_0}(\mathbf{R}^d\setminus 0)$ and every $\psi\in\mathscr{C}_{\xi_0}(\mathbf{R}^d)$ it holds

$$\psi\cdot\varphi(D)f\notin M(\omega_j,\mathscr{B}_j) \tag{4.3}$$

for some $j\in J$ (for every $j\in J$);

(3) The wave-front set $\mathrm{WF}^{\psi e,\sup}_{(\mathcal{B}_j)}(f)$ of sup-type ($\mathrm{WF}^{\psi e,\inf}_{(\mathcal{B}_j)}(f)$ of inf-type) with respect to $(\mathcal{B}_j)$, consists of all pairs (x_0,ξ_0) in $(\mathbf{R}^d\setminus 0)\times(\mathbf{R}^d\setminus 0)$ such that for every $\psi_1\in\mathscr{C}^{\mathrm{dir}}_{x_0}(\mathbf{R}^d\setminus 0)$ and every $\psi_2\in\mathscr{C}^{\mathrm{dir}}_{\xi_0}(\mathbf{R}^d\setminus 0)$ it holds

$$\psi_1\cdot\psi_2(D)f\notin M(\omega_j,\mathscr{B}_j) \tag{4.4}$$

for some $j\in J$ (for every $j\in J$).

Finally, the *global wave-front sets of sup and inf types* $\mathrm{WF}^{\sup}_{(\mathcal{B}_j)}(f)\subseteq(\mathbf{R}^d\times\mathbf{R}^d)\backslash 0$ and $\mathrm{WF}^{\inf}_{(\mathcal{B}_j)}(f)\subseteq(\mathbf{R}^d\times\mathbf{R}^d)\backslash 0$ are the sets

$$\mathrm{WF}^{\sup}_{(\mathcal{B}_j)}(f)\equiv\mathrm{WF}^{\psi,\sup}_{(\mathcal{B}_j)}(f)\cup\mathrm{WF}^{e,\sup}_{(\mathcal{B}_j)}(f)\cup\mathrm{WF}^{\psi e,\sup}_{(\mathcal{B}_j)}(f),$$

and

$$\mathrm{WF}^{\inf}_{(\mathcal{B}_j)}(f)\equiv\mathrm{WF}^{\psi,\inf}_{(\mathcal{B}_j)}(f)\cup\mathrm{WF}^{e,\inf}_{(\mathcal{B}_j)}(f)\cup\mathrm{WF}^{\psi e,\inf}_{(\mathcal{B}_j)}(f)$$

respectively.

Remark 4.2. Let $p_j,q_j\in[1,\infty]$, $\mathscr{B}_j=L^{p_j,q_j}(\mathbf{R}^{2d})$, $\omega_j(x,\xi)=\langle x,\xi\rangle^{-j}$ and let $\mathcal{B}_j$ be as in (4.1) for $j\in J=\mathbf{N}_0$. Then it follows that $\mathrm{WF}^{\psi,\sup}_{(\mathcal{B}_j)}(f)$, $\mathrm{WF}^{e,\sup}_{(\mathcal{B}_j)}(f)$ and $\mathrm{WF}^{\psi e,\sup}_{(\mathcal{B}_j)}(f)$ in Definition 4.1 with $\mathscr{B}=L^{p_j,q_j}_1$ are equal to the wave-front sets $\mathrm{WF}^{\psi}(f)$, $\mathrm{WF}^{e}(f)$ and $\mathrm{WF}^{\psi e}(f)$ in [8], respectively. In particular, it follows that $\mathrm{WF}^{\sup}_{(\mathcal{B}_j)}(f)$ is equal to the global wave-front set $\mathrm{WF}_{\mathscr{S}}(f)$, which in [8] is denoted by $\mathrm{WF}_{\mathcal{S}}(f)$.

Remark 4.3. Obviously, if $\mathscr{B}_j=\mathscr{B}$ and $\omega_j=\omega$ for every $j\in J$, then

$$\mathrm{WF}^{\psi,\sup}_{(\mathcal{B}_j)}(f)=\mathrm{WF}^{\psi,\inf}_{(\mathcal{B}_j)}(f)=\mathrm{WF}^{\psi}_{M(\omega,\mathscr{B})}(f),$$

and similarly for $\mathrm{WF}^{e,\sup}_{(\mathcal{B}_j)}(f)$ and $\mathrm{WF}^{\psi e,\sup}_{(\mathcal{B}_j)}(f)$.

The following result follows immediately from the definitions, Theorem 3.1 and its proofs. Here we let

$$\begin{aligned} (\mathcal{B}_j) \equiv (\mathcal{B}_j)_{j\in J}, \qquad &\text{and} \quad (\mathcal{C}_j) \equiv (\mathcal{C}_j)_{j\in J}, \\ \text{where} \quad \mathcal{B}_j = M(\omega_j, \mathscr{B}_j), \quad &\text{and} \qquad \mathcal{C}_j = M(\omega_j/\omega_0, \mathscr{B}_j). \end{aligned} \tag{3.1$'$}$$

Theorem 3.1$'$. *Let $r, \rho \in [0,1]$, $\mathscr{B}_j$ be translation invariant BF-spaces on $\mathbf{R}^{2d}$, $\omega_j \in \mathscr{P}(\mathbf{R}^{2d})$ for $j \in J$, $\omega_0 \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$, $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$ and let $f \in \mathscr{S}'(\mathbf{R}^d)$. Moreover, let $(\mathcal{B}_j)$ and $(\mathcal{C}_j)$ be defined as in (3.1)$'$. Then the following is true:*

(1) *if in addition $\rho > 0$, then*

$$\mathrm{WF}^{\psi,\sup}_{(\mathcal{C}_j)}(\mathrm{Op}(a)f) \subseteq \mathrm{WF}^{\psi,\sup}_{(\mathcal{B}_j)}(f) \subseteq \mathrm{WF}^{\psi,\sup}_{(\mathcal{C}_j)}(\mathrm{Op}(a)f) \cup \mathrm{Char}^{\psi}_{(\omega_0)}(a); \tag{4.5}$$

(2) *if in addition $r > 0$, then*

$$\mathrm{WF}^{e,\sup}_{(\mathcal{C}_j)}(\mathrm{Op}(a)f) \subseteq \mathrm{WF}^{e,\sup}_{(\mathcal{B}_j)}(f) \subseteq \mathrm{WF}^{e,\sup}_{(\mathcal{C}_j)}(\mathrm{Op}(a)f) \cup \mathrm{Char}^{e}_{(\omega_0)}(a); \tag{4.6}$$

(3) *if in addition $r, \rho > 0$, then*

$$\mathrm{WF}^{\psi e,\sup}_{(\mathcal{C}_j)}(\mathrm{Op}(a)f) \subseteq \mathrm{WF}^{\psi e,\sup}_{(\mathcal{B}_j)}(f) \subseteq \mathrm{WF}^{\psi e,\sup}_{(\mathcal{C}_j)}(\mathrm{Op}(a)f) \cup \mathrm{Char}^{\psi e}_{(\omega_0)}(a). \tag{4.7}$$

The same is true if the wave-front sets of sup-types are replaced by inf-types.

We note that many properties that are valid for the wave-front sets of modulation space types also hold for wave-front sets in the present Section. The following generalization of Theorem 2.6 is an immediate consequence of Theorem 3.1$'$, since $\mathrm{Char}_{(\omega_0)}(a) = \emptyset$, when a is SG-elliptic with respect to ω_0.

Theorem 3.3$'$. *Let $0 < r, \rho \le 1$, $\omega_j \in \mathscr{P}(\mathbf{R}^{2d})$ for $j \in J$, $\omega_0 \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ and let $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$ be* SG*-elliptic with respect to ω_0. Also let $\mathscr{B}_j$ be a translation invariant BF-space on $\mathbf{R}^d$ for every $j \in J$ and $(\mathcal{B}_j)$ and $(\mathcal{C}_j)$ be defined as in (3.1)$'$. If $f \in \mathscr{S}'(\mathbf{R}^d)$, then*

$$\mathrm{WF}^{\psi,\sup}_{(\mathcal{C}_j)}(\mathrm{Op}(a)f) = \mathrm{WF}^{\psi,\sup}_{(\mathcal{B}_j)}(f), \quad \mathrm{WF}^{e,\sup}_{(\mathcal{C}_j)}(\mathrm{Op}(a)f) = \mathrm{WF}^{e,\sup}_{(\mathcal{B}_j)}(f)$$

$$\mathrm{WF}^{\psi e,\sup}_{(\mathcal{C}_j)}(\mathrm{Op}(a)f) = \mathrm{WF}^{\psi e,\sup}_{(\mathcal{B}_j)}(f).$$

The same is true if the wave-front sets of sup-types are replaced by inf-types.

A combination of Remark 4.2, Theorems 2.6$'$ and 3.3$'$ and Corollary 3.4 now gives the following results concerning wave-front sets in [8]:

Corollary 4.4. *Assume that $r, \rho \in (0,1]$, and let $\omega_0 \in \mathscr{P}_{\rho,\delta}(\mathbf{R}^{2d})$, $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$, $f \in \mathscr{S}'(\mathbf{R}^d)$. Then*

$$\mathrm{WF}^{\psi}(\mathrm{Op}(a)f) \subseteq \mathrm{WF}^{\psi}(f) \subseteq \mathrm{WF}^{\psi}(\mathrm{Op}(a)f) \cup \mathrm{Char}^{\psi}_{(\omega_0)}(a),$$

$$\mathrm{WF}^{e}(\mathrm{Op}(a)f) \subseteq \mathrm{WF}^{e}(f) \subseteq \mathrm{WF}^{e}(\mathrm{Op}(a)f) \cup \mathrm{Char}^{e}_{(\omega_0)}(a),$$

$$\mathrm{WF}^{\psi e}(\mathrm{Op}(a)f) \subseteq \mathrm{WF}^{\psi e}(f) \subseteq \mathrm{WF}^{\psi e}(\mathrm{Op}(a)f) \cup \mathrm{Char}^{\psi e}_{(\omega_0)}(a).$$

Corollary 4.5. *Let $0 < r, \rho \le 1$, $\omega_0 \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ and let $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$ be* SG*-elliptic with respect to ω_0. If $f \in \mathscr{S}'(\mathbf{R}^d)$, then*

$$\mathrm{WF}^{\psi}(\mathrm{Op}(a)f) = \mathrm{WF}^{\psi}(f), \quad \mathrm{WF}^{e}(\mathrm{Op}(a)f) = \mathrm{WF}^{e}(f),$$
$$\mathrm{WF}^{\psi e}(\mathrm{Op}(a)f) = \mathrm{WF}^{\psi e}(f).$$

Corollary 4.6. *Assume that the hypotheses in Theorem 3.3′ are fulfilled, let $g \in \mathscr{S}(\mathbf{R}^d)$, and assume that $f \in \mathscr{S}'(\mathbf{R}^d)$ is a solution to the equation*

$$\mathrm{Op}(a)f = g.$$

Then $f \in \mathscr{S}$.

5. Appendix

In this Appendix we describe a proof of Proposition 1.15.

Proof of Proposition 1.15. The equivalence between (1) and (2) follows easily by letting $b(x,\xi) = \psi_1(x)\psi_2(\xi)/a(x,\xi)$, for appropriate $\psi_1 \in \mathscr{C}^{\mathrm{dir}}_{x_0}$ and $\psi_2 \in \mathscr{C}^{\mathrm{dir}}_{\xi_0}$.

(4) $\Rightarrow$ (3) is obvious in view of Remark 1.12. Assume that (3) holds. We claim that

$$|a(x,\xi)b(x,\xi)| \ge 1/2 \tag{5.1}$$

holds when

$$(x,\xi) \in \Gamma_1 \times \Gamma_2, \ |x| \ge R, \ |\xi| \ge R, \tag{5.2}$$

for some choice of conical neighbourhoods Γ_1 and Γ_2 of x_0 and ξ_0, respectively, and some $R > 0$. In fact, the assumptions imply that $ab = c + h_1$ for some $h_1 \in \mathrm{SG}^{-r,0}_{r,\rho} + \mathrm{SG}^{0,-\rho}_{r,\rho}$. By choosing R large enough and Γ_1 and Γ_2 sufficiently small conical neighbourhoods of x_0 and ξ_0, respectively, it follows that $c(x,\xi) = 1$ and $|h(x,\xi)| \le 1/2$ when (5.2) holds: this gives (5.1). Since $|b| \le C/\omega$, it follows that (1.15) is fulfilled, and (1) follows.

It remains to prove that (2) implies (4). Assume therefore that (2) is true. Let $\psi_{1,k} \in \mathscr{C}^{\mathrm{dir}}_{x_0}(\Gamma_1)$ and $\psi_{2,k} \in C^{\mathrm{dir}}_{\xi_0}(\Gamma_2)$ for $k = 1, \dots, 4$, be chosen such that

$$b_1(x,\xi) \equiv \psi_{1,1}(x)\psi_{2,1}(\xi)/a(x,\xi) \in \mathrm{SG}^{(1/\omega_0)}_{r,\rho},$$

and $\psi_{j,k} = 1$ on $\operatorname{supp}\psi_{j,k+1}$. If $c_1 = \psi_{1,1} \otimes \psi_{2,1} \in \mathrm{SG}^{0,0}_{r,\rho}$, then it follows that

$$\operatorname{Op}(b_j)\operatorname{Op}(a) = \operatorname{Op}(c_j) + \operatorname{Op}(h_j) \tag{5.3}$$

holds for $j = 1$ and some $h_1 \in \mathrm{SG}^{-r,-\rho}_{r,\rho}$.
For $j \geq 2$ we now define $\widetilde{b}_j \in \mathrm{SG}^{(1/\omega)}_{r,\rho}$ by the Neumann series

$$\operatorname{Op}(\widetilde{b}_j) = \sum_{k=0}^{j-1} (-1)^k \operatorname{Op}(\widetilde{r}_k),$$

where $\operatorname{Op}(\widetilde{r}_k) = \operatorname{Op}(h_1)^k \operatorname{Op}(b_1) \in \operatorname{Op}(\mathrm{SG}^{(\sigma_{-k\rho,-k\rho/\omega_0})}_{r,\rho})$. Then (5.3) gives

$$\begin{aligned}
\operatorname{Op}(\widetilde{b}_j)\operatorname{Op}(a) &= \sum_{k=0}^{j-1} (-1)^k \operatorname{Op}(h_1)^k \operatorname{Op}(b_1)\operatorname{Op}(a) \\
&= \sum_{k=0}^{j-1} (-1)^k \operatorname{Op}(h_1)^k (\operatorname{Op}(c_1) + \operatorname{Op}(h_1)) \\
&= \operatorname{Op}(c_1) + \operatorname{Op}(\widetilde{h}_{1,j}) + \operatorname{Op}(\widetilde{h}_{2,j}),
\end{aligned} \tag{5.4}$$

where

$$\operatorname{Op}(\widetilde{h}_{1,j}) = (-1)^j \operatorname{Op}(h_1)^j \in \operatorname{Op}(\mathrm{SG}^{-jr,-j\rho}_{r,\rho}) \tag{5.5}$$

and

$$\operatorname{Op}(\widetilde{h}_{2,j}) = -\sum_{k=1}^{j-1} (-1)^k \operatorname{Op}(h_1)^k \operatorname{Op}(1-c_1) \in \operatorname{Op}(\mathrm{SG}^{0,0}_{r,\rho}).$$

By asymptotic expansions it follows that

$$\operatorname{Op}(\widetilde{h}_{2,j}) = -\sum_{k=1}^{j-1} (-1)^k \operatorname{Op}(1-c_1)\operatorname{Op}(h_1)^k + \operatorname{Op}(\widetilde{h}_{3,j}) + \operatorname{Op}(\widetilde{h}_{4,j}), \tag{5.6}$$

for some $\widetilde{h}_{3,j} \in \mathrm{SG}^{-r,-\rho}_{r,\rho}$ which is equal to zero on $\operatorname{supp} c_1$ and $\widetilde{h}_{4,j} \in \mathrm{SG}^{-jr,-j\rho}_{r,\rho}$. Now let c, b_j and r_k be defined by the formulae

$$c(x,\xi) = \psi_{1,3}(x)\psi_{2,3}(\xi), \quad \operatorname{Op}(b_j) = \operatorname{Op}(c)\operatorname{Op}(\widetilde{b}_j) \in \operatorname{Op}(\mathrm{SG}^{(1/\omega_0)}_{r,\rho}),$$

$$\operatorname{Op}(r_k) = \operatorname{Op}(c)\operatorname{Op}(\widetilde{r}_k) \in \operatorname{Op}(\mathrm{SG}^{(\sigma_{-k\rho,-k\rho/\omega_0})}_{r,\rho}).$$

Then

$$\operatorname{Op}(b_j) = \sum_{k=0}^{j-1} (-1)^k \operatorname{Op}(r_k)$$

and (5.4)–(5.6) give

$$\operatorname{Op}(b_j)\operatorname{Op}(a) = \operatorname{Op}(c)\operatorname{Op}(c_1) + \operatorname{Op}(c)\operatorname{Op}(\widetilde{h}_{1,j})$$

$$-\sum_{k=1}^{j-1}(-1)^k \operatorname{Op}(c)\operatorname{Op}(1-c_1)\operatorname{Op}(h_1)^k + \operatorname{Op}(c)\operatorname{Op}(\widetilde{h}_{3,j}) + \operatorname{Op}(c)\operatorname{Op}(\widetilde{h}_{4,j}).$$

Since $c_1 = 1$ and $\widetilde{h}_{3,j} = 0$ on $\operatorname{supp} c$, and every element of $\operatorname{Op}(\mathrm{SG}^{-\infty,-\infty}_{r,\rho})$ maps continuously $\mathscr{S}'$ to $\mathscr{S}$, we find

$$\operatorname{Op}(c)\operatorname{Op}(c_1) = \operatorname{Op}(c) \mod \operatorname{Op}(\mathscr{S}),$$

$$\operatorname{Op}(c)\operatorname{Op}(\widetilde{h}_{1,j}) \in \operatorname{Op}(\mathrm{SG}^{-jr,-j\rho}_{r,\rho}),$$

$$\sum_{k=1}^{j-1}(-1)^k \operatorname{Op}(c)\operatorname{Op}(1-c_1)\operatorname{Op}(h_1)^k \in \operatorname{Op}(\mathscr{S}),$$

$$\operatorname{Op}(c)\operatorname{Op}(\widetilde{h}_{3,j}) \in \operatorname{Op}(\mathscr{S}),$$

and

$$\operatorname{Op}(c)\operatorname{Op}(\widetilde{h}_{4,j}) \in \operatorname{Op}(\mathrm{SG}^{-jr,-j\rho}_{r,\rho}).$$

Hence, (5.3) follows for $c_j = c$ and some $h_j \in \mathrm{SG}^{-jr,-j\rho}_{r,\rho}$. By choosing $b \in \mathrm{SG}^{(1/\omega)}_{r,\rho}$ such that

$$b \sim \sum r_k,$$

the argument above shows that $\operatorname{Op}(b)\operatorname{Op}(a) = \operatorname{Op}(c) + \operatorname{Op}(h)$, with $h \in \mathrm{SG}^{-\infty,-\infty}_{r,\rho} = \mathscr{S}$, and (4) follows. The proof is complete. □

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Dipartimento di Matematica, Università di Torino, Italy
E-mail address: sandro.coriasco@unito.it

Department of Mathematics and Systems Engineering, Växjö University, Sweden
E-mail address: karoline.johansson@vxu.se

Department of Mathematics and Systems Engineering, Växjö University, Sweden
E-mail address: joachim.toft@vxu.se